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Reductions of N-wave interactions related to low-rank simple Lie algebras: I. $\mathbb{Z}_2$-reductions

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Abstract

The analysis and the classification of all reductions for the nonlinear evolution equations solvable by the inverse scattering method is an interesting and still open problem. We show how the second-order reductions of the N-wave interactions related to low-rank simple Lie algebras $\mathfrak{g}$ can be embedded also in the Weyl group of $\mathfrak{g}$. This allows us to display along with the well known ones a number of new types of integrable N-wave systems. Some of the reduced systems find applications to nonlinear optics.

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1. Introduction

It is well known that the N-wave equations [1–6]

$$\mathrm{i}[J, Q_t] - \mathrm{i}[I, Q_x] + [[I, Q], [J, Q]] = 0 \tag{1.1}$$

are solvable by the inverse scattering method (ISM) [4, 5] applied to the generalized system of Zakharov–Shabat type [4, 7, 8]:

$$L(\lambda)\Psi(x, t, \lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + [J, Q(x, t)] - \lambda J\right)\Psi(x, t, \lambda) = 0 \qquad J \in \mathfrak{h} \tag{1.2}$$

$$Q(x, t) = \sum_{\alpha\in\Delta} Q_\alpha(x, t)E_\alpha \equiv \sum_{\alpha\in\Delta_+}(q_\alpha(x, t)E_\alpha + p_\alpha(x, t)E_{-\alpha}) \in \mathfrak{g}\backslash\mathfrak{h} \tag{1.3}$$

where $Q(x, t)$ and J take values in the simple Lie algebra $\mathfrak{g}$. Here $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, Δ (resp. Δ_+) is the system of roots (resp., the system of positive roots) of $\mathfrak{g}$; E_α are the root vectors of the simple Lie algebra $\mathfrak{g}$. Indeed, equation (1.1) is the compatibility condition

$$[L(\lambda), M(\lambda)] = 0 \tag{1.4}$$

where

$$M(\lambda)\Psi(x, t, \lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + [I, Q(x, t)] - \lambda I\right)\Psi(x, t, \lambda) = 0 \qquad I \in \mathfrak{h}. \tag{1.5}$$

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Here and below $r = \operatorname{rank} \mathfrak{g}$, and $\vec{a}, \vec{b} \in \mathbb{E}^r$ are the vectors corresponding to the Cartan elements $J, I \in \mathfrak{h}$.

The inverse scattering problem for (1.2) with real-valued J [1] was reduced to a Riemann–Hilbert problem (RHP) for the (matrix-valued) fundamental analytic solution of (1.2) [4, 7, 8]; the action-angle variables for the N-wave equations with $\mathfrak{g} \simeq sl(n)$ were obtained in the preprint [1], see also [9]. Most of these results were derived first for the simplest non-trivial case when J has pair-wise distinct real eigenvalues.

However, often the reduction conditions require that J be complex-valued, see [8–10]. Then the construction of the fundamental analytic solutions (FAS) and the solution of the corresponding inverse scattering problem for (1.2) becomes more difficult [11, 12].

The interpretation of the ISM as a generalized Fourier transform and the expansions over the 'squared solutions' of (1.2) were derived in [8] for real J and in [11] for complex J. This interpretation allows one also to prove that all N-wave type equations are Hamiltonian and possess a hierarchy of pair-wise compatible Hamiltonian structures [8, 11] $\{H^{(k)}, \Omega^{(k)}\}$, $k = 0, \pm 1, \pm 2, \ldots$.

Indeed, as a phase space $\mathcal{M}_{ph}$ of these equations one can choose the space spanned by the complex-valued functions $\{q_\alpha, p_\alpha, \alpha \in \Delta_+\}$, $\dim_{\mathbb{C}} \mathcal{M}_{ph} = |\Delta|$. The corresponding nonlinear evolution equation (NLEE) as, for example, (1.1) and its higher analogues can be formally written down as Hamiltonian equations of motion:

$$\Omega^{(k)}(Q_t, \cdot) = \mathrm{d}H^{(k)}(\cdot) \qquad k = 0, \pm 1, \pm 2, \ldots \tag{1.6}$$

where both $\Omega^{(k)}$ and $H^{(k)}$ are complex-valued. The simplest Hamiltonian formulation of (1.1) is given by $\{H^{(0)}, \Omega^{(0)}\}$ where $H^{(0)} = H_0 + H_{\mathrm{I}}$ and

$$H_0 = \frac{c_0}{2\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x\ \langle Q, [I, Q_x] \rangle = \sum_{\alpha \in \Delta_+} H_0(\alpha) \tag{1.7}$$

$$H_0(\alpha) = \mathrm{i}c_0 \int_{-\infty}^{\infty} \mathrm{d}x\ \frac{(\vec{b}, \alpha)}{(\alpha, \alpha)} (q_\alpha p_{\alpha,x} - q_{\alpha,x} p_\alpha) \tag{1.8}$$

$$\begin{gathered} H_{\mathrm{I}} = \frac{c_0}{3} \int_{-\infty}^{\infty} \mathrm{d}x\ \langle [J, Q], [Q, [I, Q]] \rangle = \sum_{[\alpha,\beta,\gamma] \in \mathcal{M}} \omega_{\beta,\gamma} H(\alpha, \beta, \gamma) \\ H(\alpha, \beta, \gamma) = c_0 \int_{-\infty}^{\infty} \mathrm{d}x\ (q_\alpha p_\beta p_\gamma - p_\alpha q_\beta q_\gamma) \quad \omega_{\beta,\gamma} = \frac{4N_{\beta,\gamma}}{(\alpha, \alpha)} \det \begin{pmatrix} (\vec{a}, \beta) & (\vec{b}, \gamma) \\ (\vec{a}, \beta) & (\vec{b}, \gamma) \end{pmatrix} \end{gathered} \tag{1.9}$$

and the symplectic form $\Omega^{(0)}$ is equivalent to a canonical one

$$\Omega^{(0)} = \frac{\mathrm{i}c_0}{2} \int_{-\infty}^{\infty} \mathrm{d}x\ \langle [J, \delta Q(x,t)] \underset{,}{\wedge} \delta Q(x,t) \rangle = \sum_{\alpha \in \Delta_+} \Omega^{(0)}(\alpha) \tag{1.10}$$

$$\Omega^{(0)}(\alpha) = \mathrm{i}c_0 \frac{2(\vec{a}, \alpha)}{(\alpha, \alpha)} \int_{-\infty}^{\infty} \mathrm{d}x\ \delta q_\alpha(x,t) \wedge \delta p_\alpha(x,t). \tag{1.11}$$

Here c_0 is a constant to be explained below, $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}$ and the triple $[\alpha, \beta, \gamma]$ belongs to $\mathcal{M}$ if $\alpha, \beta, \gamma \in \Delta_+$ and $\alpha = \beta + \gamma$; $N_{\beta,\gamma}$ is defined in (2.13), (2.14) below.

The Hamiltonian of the N-wave equations and their higher analogues $H^{(k)}$ depend analytically on q_α, p_α. That allows one to rewrite equation (1.6) as a standard Hamiltonian equation with real-valued $\Omega^{(k)}$ and $H^{(k)}$. The phase space then is viewed as the manifold of real-valued functions $\{\operatorname{Re} q_\alpha, \operatorname{Re} p_\alpha, \operatorname{Im} q_\alpha, \operatorname{Im} p_\alpha\}$, $\alpha \in \Delta_+$, so $\dim_{\mathbb{R}} \mathcal{M}_{ph} = 2|\Delta|$. Such treatment is rather formal and we do not explain it in further detail here.

Another better known way to make $\Omega^{(k)}$ and $H^{(k)}$ real is to impose reduction on them involving complex or Hermitian conjugation; below we list several types of such reductions. Obviously we can multiply both sides of (1.6) by the same constant c_k. We will use this freedom below and whenever possible will adjust the constant c_0 (or c_k) in such a way that both $\Omega^{(0)}$, $H^{(0)}$ (or $\Omega^{(k)}$, $H^{(k)}$) are real.

Physically to each term $H(\alpha, \beta, \gamma)$ we relate part of a wave-decay diagram which shows how the α-wave decays into β- and γ-waves. In other words we assign to each root α a wave with wavenumber k_α and frequency ω_α. Each of the elementary decays preserves them, i.e.

$$k_\alpha = k_\beta + k_\gamma \qquad \omega(k_\alpha) = \omega(k_\beta) + \omega(k_\gamma).$$

Our aim is to investigate all inequivalent $\mathbb{Z}_2$ reductions of the N-wave type equations related to the low-rank simple Lie algebras. Thus we exhibit new examples of integrable N-wave type interactions, some of which have applications to physics.

From the definition of the reduction group G_R introduced by Mikhailov in [10] and further developed in [13–15] it is natural that we have to have realizations of G_R as (i) finite subgroup of the group Aut $(\mathfrak{g})$ of automorphisms of the algebra $\mathfrak{g}$ and (ii) finite subgroup of the conformal mappings on the complex λ-plane. We also impose the natural restriction that the reduction preserves the form of the Lax operator (1.2). In particular, that means that we have to limit ourselves only to those elements of Aut $(\mathfrak{g})$ that preserve the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This condition narrows our choice to $V_0 \otimes \mathrm{Ad}(\mathfrak{h}) \otimes W(\mathfrak{g})$ where $W(\mathfrak{g})$ is the Weyl group of $\mathfrak{g}$ and V_0 is the group of external automorphisms of $\mathfrak{g}$. As a result we find that the reduction is sensitive to the way G_R is embedded into Aut $(\mathfrak{g})$.

We start with the $\mathbb{Z}_2$-reductions which provide the richest class of interesting examples and display a number of non-trivial and inequivalent reductions for the N-wave type equations.

In section 2 we briefly outline the main idea of the reduction group [10]. In section 3 we introduce convenient notation and describe how generic $\mathbb{Z}_2$-reductions act on $H^{(0)}$ and $\Omega^{(0)}$. We also list the properties of the Weyl groups for the algebras $\boldsymbol{A}_k$, $\boldsymbol{B}_k$, $\boldsymbol{C}_k$, $k = 2, 3$ and $\boldsymbol{G}_2$. The main attention here is paid to their equivalence classes. Obviously if two reductions generated by two different automorphisms A and A' are inequivalent then A and A' must belong to different equivalence classes of $W(\mathfrak{g})$.

The list of the inequivalent reductions is displayed in section 4. For each of the cases we list the restrictions imposed on the potential Q and the Cartan elements J and I. Whenever the reduction preserves the simplest Hamiltonian structure (1.7), (1.9), (1.10) we write down only the Hamiltonian. In the cases when the reduction makes the simplest Hamiltonian structure degenerate we write down the system of equations.

Whenever G_R acts on $\mathrm{i}U(x, \lambda)$ as Cartan involution the result of the reduction is to get a real form of the algebra $\mathfrak{g}$. These cases are discussed in section 5.

In section 6 we formulate the effect of G_R on the scattering data of the Lax operator. In section 7 these facts are used to explain how in certain cases the reduction makes 'half' of the symplectic forms and 'half' of the Hamiltonians in the hierarchy degenerate. The paper finishes with several conclusions and two appendices which contain some subsidiary facts about the root systems of the Lie algebras (appendix A) and the typical interaction terms in H_I (appendix B).

This paper is an extended exposition of a part of our reports [16] with some misprints corrected.

2. Preliminaries and general approach

The main idea underlying Mikhailov's reduction group [10] is to impose algebraic restrictions on the Lax operators L and M which will be automatically compatible with the corresponding equations of motion (1.4). Due to the purely Lie-algebraic nature of the Lax representation (1.4) this is most naturally done by embedding the reduction group as a subgroup of Aut $\mathfrak{g}$—the group of automorphisms of $\mathfrak{g}$. Obviously, to each reduction imposed on L and M there will correspond a reduction of the space of fundamental solutions $\mathfrak{S}_\Psi \equiv \{\Psi(x,t,\lambda)\}$ of (1.2) and (1.5).

Some of the simplest $\mathbb{Z}_2$-reductions of N-wave systems have been known for a long time (see [10]) and are related to external automorphisms of $\mathfrak{g}$ and $\mathfrak{G}$, namely

$$C_1(\Psi(x,t,\lambda)) = A_1\Psi^\dagger(x,t,\kappa_1(\lambda))A_1^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda) \qquad \kappa_1(\lambda) = \pm\lambda^* \tag{2.1}$$

where A_1 belongs to the Cartan subgroup of the group $\mathfrak{G}$:

$$A_1 = \exp(\pi \mathrm{i} H_1) \tag{2.2}$$

and $H_1 \in \mathfrak{h}$ is such that $\alpha(H_1) \in \mathbb{Z}$ for all roots $\alpha \in \Delta$ in the root system Δ of $\mathfrak{g}$. Note that the reduction condition relates the fundamental solution $\Psi(x,t,\lambda) \in \mathfrak{G}$ to a fundamental solution $\tilde{\Psi}(x,t,\lambda)$ of (1.2) and (1.5) which in general differs from $\Psi(x,t,\lambda)$.

Another class of $\mathbb{Z}_2$ reductions are related to external automorphisms of the type

$$C_2(\Psi(x,t,\lambda)) = A_2\Psi^T(x,t,\kappa_2(\lambda))A_2^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda) \qquad \kappa_2(\lambda) = \pm\lambda \tag{2.3}$$

where A_2 is again of the form (2.2). The best known examples of NLEE obtained with the reduction (2.3) are the sine–Gordon and the MKdV equations which are related to $\mathfrak{g} \simeq sl(2)$. For higher-rank algebras such reductions to our knowledge have not been studied. Generically, reductions of type (2.3) lead to degeneration of the canonical Hamiltonian structure, i.e. $\Omega^{(0)} \equiv 0$; then we need to use some of the higher Hamiltonian structures (see [10, 11]) for proving their complete integrability.

In fact, the reductions (2.1) and (2.3) provide us examples when the reduction is obtained with the combined use of external and inner automorphisms.

As well as (2.2), (2.1) one may also use reductions with inner automorphisms:

$$C_3(\Psi(x,t,\lambda)) = A_3\Psi^*(x,t,\kappa_1(\lambda))A_3^{-1} = \tilde{\Psi}(x,t,\lambda) \tag{2.4}$$

and

$$C_4(\Psi(x,t,\lambda)) = A_4\Psi(x,t,\kappa_2(\lambda))A_4^{-1} = \tilde{\Psi}(x,t,\lambda). \tag{2.5}$$

Since our aim is to preserve the form of the Lax pair we limit ourselves to automorphisms preserving the Cartan subalgebra $\mathfrak{h}$. This condition is obviously fulfilled if A_k, $k = 1,\ldots,4$ is in the form (2.2). Another possibility is to choose $A_1,\ldots,A_4$ so that they correspond to Weyl group automorphisms.

In fact (2.1) and (2.3) are related to external automorphisms only if $\mathfrak{g}$ is from the $\boldsymbol{A}_r$ series. For the $\boldsymbol{B}_r$, $\boldsymbol{C}_r$ and $\boldsymbol{D}_r$ series (2.1) is equivalent to an inner automorphism (2.4) with the special choice for the Weyl group element w_0 which maps all highest-weight vectors into the corresponding lowest-weight vectors (see remark (1)). Finally, $\mathbb{Z}_2$ reductions of the form (2.1) in fact restrict us to the corresponding real form of the algebra $\mathfrak{g}$.

2.1. The reduction group

The reduction group G_R is a finite group which preserves the Lax representation (1.4), i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations: (i) $G_R \subset \mathrm{Aut}\,\mathfrak{g}$ and (ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [10]:

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda) \qquad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda) \tag{2.6}$$

where $C_k \in \mathrm{Aut}\,\mathfrak{g}$ and $\Gamma_k(\lambda) \in \mathrm{Conf}\,\mathbb{C}$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$. In all the cases below $N_k = 2$ and the reduction group is isomorphic to $\mathbb{Z}_2$.

More specifically, the automorphisms C_k, $k = 1, \dots, 4$ listed above lead to the following reductions for the matrix-valued functions

$$U(x,t,\lambda) = [J, Q(x,t)] - \lambda J \qquad V(x,t,\lambda) = [I, Q(x,t)] - \lambda I \tag{2.7}$$

of the Lax representation:

$$\begin{aligned}
&(1)\quad C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda) && C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda) && (2.8a)\\
&(2)\quad C_2(U^T(\kappa_2(\lambda))) = -U(\lambda) && C_2(V^T(\kappa_2(\lambda))) = -V(\lambda) && (2.8b)\\
&(3)\quad C_3(U^*(\kappa_1(\lambda))) = -U(\lambda) && C_3(V^*(\kappa_1(\lambda))) = -V(\lambda) && (2.8c)\\
&(4)\quad C_4(U(\kappa_2(\lambda))) = U(\lambda) && C_4(V(\kappa_2(\lambda))) = V(\lambda). && (2.8d)
\end{aligned}$$

2.2. *Finite groups*

The condition (2.6) is obviously compatible with the group action. Therefore it is enough to ensure that (2.6) is fulfilled for the generating elements of G_R.

In fact (see [17]) every finite group G is determined uniquely by its generating elements g_k and genetic code: e.g.,

$$g_k^{N_k} = 1\!\!1 \qquad (g_j g_k)^{N_{jk}} = 1\!\!1 \qquad N_k, N_{jk} \in \mathbb{Z}. \tag{2.9}$$

For example, the cyclic $\mathbb{Z}_N$ and the dihedral $\mathbb{D}_N$ groups have as genetic codes

$$g^N = 1\!\!1 \qquad N \geqslant 2 \qquad \text{for} \quad \mathbb{Z}_N \tag{2.10}$$

and

$$g_1^2 = g_2^2 = (g_1 g_2)^N = 1\!\!1 \qquad N \geqslant 2 \qquad \text{for} \quad \mathbb{D}_N. \tag{2.11}$$

2.3. *Cartan–Weyl basis and Weyl group*

Here we fix the notations and the normalization conditions for the Cartan–Weyl generators of $\mathfrak{g}$. We introduce $h_k \in \mathfrak{h}$, $k = 1, \dots, r$ and E_α, $\alpha \in \Delta$ where $\{h_k\}$ are the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$. As well as h_k we also introduce

$$H_\alpha = \frac{2}{(\alpha,\alpha)} \sum_{k=1}^{r} (\alpha, e_k) h_k \qquad \alpha \in \Delta \tag{2.12}$$

where (α, e_k) is the scalar product in the root space $\mathbb{E}^r$ between the root α and e_k. The commutation relations are given by [18, 21]

$$\begin{aligned}
&[h_k, E_\alpha] = (\alpha, e_k) E_\alpha \qquad [E_\alpha, E_{-\alpha}] = H_\alpha \\
&[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & \text{for} \quad \alpha+\beta \in \Delta \\ 0 & \text{for} \quad \alpha+\beta \notin \Delta \cup \{0\}. \end{cases}
\end{aligned} \tag{2.13}$$

We will denote by $\vec{a} = \sum_{k=1}^r a_k e_k$ the r-dimensional vector dual to $J \in \mathfrak{h}$; obviously $J = \sum_{k=1}^r a_k h_k$. If J is a regular real element in $\mathfrak{h}$ then without restrictions we may use it to introduce an ordering in Δ. Namely, we will say that the root $\alpha \in \Delta_+$ is positive (negative) if $(\alpha, \vec{a}) > 0$ ($(\alpha, \vec{a}) < 0$ respectively). The normalization of the basis is determined by

$$\begin{aligned}
&E_{-\alpha} = E_\alpha^T \qquad \langle E_{-\alpha}, E_\alpha \rangle = \frac{2}{(\alpha,\alpha)} \\
&N_{-\alpha,-\beta} = -N_{\alpha,\beta} \qquad N_{\alpha,\beta} = \pm(p+1)
\end{aligned} \tag{2.14}$$

where the integer $p \geqslant 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \ldots, p$ and $\alpha + (p+1)\beta \notin \Delta$. The root system Δ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections S_α; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha, \alpha)}\alpha \qquad \alpha \in \Delta. \tag{2.15}$$

All Weyl reflections S_α form a finite group $W_\mathfrak{g}$ known as the Weyl group. One may introduce in a natural way an action of the Weyl group on the Cartan–Weyl basis, namely:

$$\begin{aligned} S_\alpha(H_\beta) &\equiv A_\alpha H_\beta A_\alpha^{-1} = H_{S_\alpha \beta} \\ S_\alpha(E_\beta) &\equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{S_\alpha\beta} \qquad n_{\alpha,\beta} = \pm 1. \end{aligned} \tag{2.16}$$

It is also well known (see [22]) that the matrices A_α are given (up to a factor from the Cartan subgroup) by

$$A_\alpha = \mathrm{e}^{E_\alpha}\mathrm{e}^{-E_{-\alpha}}\mathrm{e}^{E_\alpha} H_A \tag{2.17}$$

where H_A is a conveniently chosen element from the Cartan subgroup such that $H_A^2 = \mathbb{1}$. The formula (2.17) and the explicit form of the Cartan–Weyl basis in the typical representation will be used in calculating the reduction condition following from (2.6).

2.4. Graded Lie algebras

One of the important notions in constructing integrable equations and their reductions is the one of graded Lie algebra and Kac–Moody algebras [18]. The standard construction is based on a finite-order automorphism $C \in \operatorname{Aut} \mathfrak{g}$, $C^N = \mathbb{1}$. Obviously, the eigenvalues of C are ω^k, $k = 0, 1, \ldots, N-1$, where $\omega = \exp(2\pi \mathrm{i}/N)$. To each eigenvalue there corresponds a linear subspace $\mathfrak{g}^{(k)} \subset \mathfrak{g}$ determined by

$$\mathfrak{g}^{(k)} \equiv \left\{X : X \in \mathfrak{g},\ C(X) = \omega^k X\right\}. \tag{2.18}$$

Obviously, $\mathfrak{g} = \bigoplus_{k=0}^{N-1} \mathfrak{g}^{(k)}$ and the grading condition holds

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(n)}\right] \subset \mathfrak{g}^{(k+n)} \tag{2.19}$$

where $k+n$ is taken modulo N. Thus to each pair $\{\mathfrak{g}, C\}$ one can relate an infinite-dimensional algebra of Kac–Moody type $\hat{\mathfrak{g}}_C$ whose elements are

$$X(\lambda) = \sum_k X_k \lambda^k \qquad X_k \in \mathfrak{g}^{(k)}. \tag{2.20}$$

The series in (2.20) must contain only a finite number of negative (positive) powers of λ and $\mathfrak{g}^{(k+N)} \equiv \mathfrak{g}^{(k)}$. This construction is a most natural one for Lax pairs; we see that due to the grading condition (2.19) we can always impose a reduction on $L(\lambda)$ and $M(\lambda)$ such that both $U(x,t,\lambda)$ and $V(x,t,\lambda) \in \hat{\mathfrak{g}}_C$. So one of the generating elements of G_R will be used for introducing a grading in $\mathfrak{g}$; then the reduction condition (2.6) gives

$$U_0, V_0 \in \mathfrak{g}^{(0)} \qquad I, J \in \mathfrak{g}^{(1)} \cap \mathfrak{h}. \tag{2.21}$$

If in particular $N = 2$, the automorphism C has the form (2.1) and $\kappa(\lambda) = \lambda^*$, then all X_k in (2.20) must be elements of the real form of $\mathfrak{g}$ defined by C. We will pay special attention to this situation in section 5.

A possible second reduction condition will enforce additional constraints on U_0, V_0 and J, I.

2.5. Realizations of $G_R \subset \mathrm{Aut}\,\mathfrak{g}$

It is well known that $\mathrm{Aut}\,\mathfrak{g} \equiv V \otimes \mathrm{Aut}_0\mathfrak{g}$ where V is the group of external automorphisms (the symmetry group of the Dynkin diagram) and $\mathrm{Aut}_0\mathfrak{g}$ is the group of inner automorphisms. Since we start with $I, J \in \mathfrak{h}$ it is natural to consider only those inner automorphisms that preserve the Cartan subalgebra $\mathfrak{h}$. Then $\mathrm{Aut}_0\mathfrak{g} \simeq \mathrm{Ad}_H \otimes W$ where Ad_H is the group of similarity transformations with elements from the Cartan subgroup

$$\mathrm{Ad}_C X = CXC^{-1} \qquad C = \exp\left(\frac{2\pi \mathrm{i} H_{\vec{c}}}{N}\right) \quad X \in \mathfrak{g} \tag{2.22}$$

and W is the Weyl group of $\mathfrak{g}$. Its action on the Cartan–Weyl basis was described in (2.16). From (2.13) one easily finds

$$CH_\alpha C^{-1} = H_\alpha \qquad CE_\alpha C^{-1} = \mathrm{e}^{2\pi \mathrm{i}(\alpha,\vec{c})/N} E_\alpha \tag{2.23}$$

where $\vec{c} \in \mathbb{E}^r$ is the vector corresponding to $H_{\vec{c}} \in \mathfrak{h}$ in (2.22). Then the condition $C^N = 1\!\!1$ means that $(\alpha, \vec{c}) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Obviously, $H_{\vec{c}}$ must be chosen so that $\vec{c} = \sum_{k=1}^{r} 2c_k\omega_k/(\alpha_k, \alpha_k)$ where ω_k are the fundamental weights of $\mathfrak{g}$ and c_k are integer. In the examples below we use several possibilities by choosing C_k as appropriate compositions of elements from V, $\mathrm{Ad}_{\mathfrak{H}}$ and W. In fact if $\mathfrak{g} \simeq \boldsymbol{G}_2$ or belongs to $\boldsymbol{B}_r$ or $\boldsymbol{C}_r$ series then $V \equiv 1\!\!1$.

2.6. Realizations of $G_R \subset \mathrm{Conf}\,\mathbb{C}$

Generically, each element $g_k \in G_R$ maps λ into a fraction–linear function of λ. Such action, however, is appropriate for a more general class of Lax operators which are fraction–linear functions of λ. Since our Lax operators are linear in λ then we have the following possibilities for $\mathbb{Z}_2$:

$$\begin{aligned} &\Gamma_1(\lambda) = a_0 + \eta\lambda \qquad && \eta = \pm 1 && \\ &\Gamma_2(\lambda) = b_0 + \epsilon\lambda^* \qquad && \epsilon = \pm 1 \qquad && b_0 + \epsilon b_0^* = 0. \end{aligned} \tag{2.24}$$

In the examples below $a_0 = b_0 = 0$.

3. Inequivalent reductions

We will consider two substantially different types of reductions (2.1). The first and best known type of $\mathbb{Z}_2$-reductions corresponds to inner automorphisms C_j from the Cartan subgroup which have the form (2.22) with $N = 2$.

For each of these reductions we will describe the structure of $\Omega^{(0)}$, H_0 and H_{I}. To make the notation more convenient we introduce $\left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}$ and $\mathcal{H}_{\mathrm{I}}(\alpha)$ as follows:

$$\left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\} = \omega_{\beta,\gamma} H(\alpha, \beta, \gamma) = c_0\omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (Q_\alpha Q_{-\beta} Q_{-\gamma} - Q_{-\alpha} Q_\beta Q_\gamma) \tag{3.1}$$

$$\mathcal{H}_{\mathrm{I}}(\alpha) = \sum_{[\beta,\gamma]\in\mathcal{M}_\alpha} \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\} \tag{3.2}$$

where Q_α are introduced in (1.3) and $\mathcal{M}_\alpha$ is the set of pairs of roots $[\beta, \gamma]$ such that $\beta + \gamma = \alpha$, see appendix A. In the last summation we do not require β and γ to be positive. The explicit expression for $\left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}$ allows us to check that

$$\left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\} = \left\{ \begin{matrix} \gamma \\ -\beta, \alpha \end{matrix} \right\} = \left\{ \begin{matrix} -\alpha \\ -\beta, -\gamma \end{matrix} \right\}. \tag{3.3}$$

In proving (3.3) we used (2.14), (2.16) and the properties of the structure constants $N_{\beta,\gamma}$ of the Chevalley basis; namely, if $\alpha - \beta - \gamma = 0$ then

$$\frac{N_{\alpha,-\beta}}{(\gamma,\gamma)} = \frac{N_{-\beta,-\gamma}}{(\alpha,\alpha)} = \frac{N_{-\gamma,\alpha}}{(\beta,\beta)} = -\frac{N_{\beta,\gamma}}{(\alpha,\alpha)} \tag{3.4}$$

and as a consequence

$$\omega_{\alpha,-\beta} = \omega_{-\beta,-\gamma} = \omega_{-\gamma,\alpha} = -\omega_{\beta,\gamma}. \tag{3.5}$$

From (1.9) it also follows that

$$H_{\mathrm{I}} = \sum_{[\alpha,\beta,\gamma]\in\mathcal{M}} \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\} = \frac{1}{3} \sum_{\alpha\in\Delta_+} \mathcal{H}_{\mathrm{I}}(\alpha). \tag{3.6}$$

Indeed, using (3.3) we find that each triple $\left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}$ is equal to $\left\{ \begin{matrix} \tilde{\alpha} \\ \tilde{\beta}, \tilde{\gamma} \end{matrix} \right\}$ where the triple of roots $[\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}] \in \mathcal{M}$.

3.1. Reductions with Cartan subgroup elements

For the first type, the action of the reduction group on the Cartan–Weyl basis is given by (2.23) and the corresponding constraints on q_α, p_α have the form

$$(1) \quad p_\alpha = -\eta s_\alpha q_\alpha^* \qquad s_\alpha = \mathrm{e}^{-\pi \mathrm{i}(\vec{c},\alpha)} \tag{3.7a}$$

$$(2) \quad p_\alpha = s_\alpha q_\alpha \qquad \eta = -1 \tag{3.7b}$$

$$(3) \quad q_\alpha = \eta s_\alpha q_\alpha^* \qquad p_\alpha = \eta s_\alpha p_\alpha^* \tag{3.7c}$$

$$(4) \quad q_\alpha = s_\alpha q_\alpha \qquad p_\alpha = s_\alpha p_\alpha \qquad \eta = 1. \tag{3.7d}$$

Let us describe how each of these constraints simplify the Hamiltonian $H^{(0)} = H_0 + H_{\mathrm{I}}$ and the symplectic form $\Omega^{(0)}$. We can write them down in the form

$$H_0 = \sum_{\alpha\in\Delta_+} s_\alpha H_{0*}(\alpha) \qquad H_{\mathrm{I}} = \sum_{\alpha\in\Delta_+} s_\alpha H_{\mathrm{I}}(\alpha) \qquad \Omega^{(0)} = \sum_{\alpha\in\Delta_+} s_\alpha \Omega_*^{(0)}(\alpha). \tag{3.8}$$

For case (1) we easily find

$$H_{0*}(\alpha) = -\mathrm{i}\eta c_0 \frac{(b,\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x \left(q_\alpha q_{\alpha,x}^* - q_{\alpha,x} q_\alpha^* \right) \tag{3.9}$$

$$H_{\mathrm{I}}(\alpha) = \sum_{\beta+\gamma=\alpha} c_0 \omega_{\beta,\gamma} s_\alpha \int_{-\infty}^{\infty} \mathrm{d}x \left(q_\alpha q_\beta^* q_\gamma^* + \eta q_\alpha^* q_\beta q_\gamma \right) \tag{3.10}$$

$$\Omega_*^{(0)}(\alpha) = -\mathrm{i}\eta c_0 \frac{2(a,\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x \, \delta q_\alpha \wedge \delta q_\alpha^*. \tag{3.11}$$

We have $|\Delta_+|$ complex-valued fields q_α and the above expressions for $\Omega^{(0)}$ and H_0 show that q_α is dynamically conjugated to q_α^*.

For case (2) we consider only $\eta = -1$; the choice $\eta = 1$ means that $\vec{a} = -\vec{a}$ and as a consequence we have $J = 0$, i.e. no N-wave equations are possible for $\eta = 1$. For $\eta = -1$ we have $|\Delta_+|$ complex valued fields q_α which up to a sign coincide with p_α. Then $H_0(\alpha)$, $\Omega^{(0)}(\alpha)$ and H_{I} become identically zero. The corresponding set of equations is nontrivial but is does not allow a canonical Hamiltonian formulation. However, it allows a Hamiltonian description using other members in the hierarchy of Hamiltonian structures.

In case (3) we have $2|\Delta_+|$ 'real'-valued fields q_α and p_α. (Here and below we count as 'real' also the fields that are in fact purely imaginary.) The formulae for $H_0(\alpha)$, $H_{\mathrm{I}}(\alpha)$ and

$\Omega^{(0)}(\alpha)$ (1.7)–(1.11) do not change, only each of the summands in them becomes real due to the fact that all fields q_α and p_α are now simultaneously either real or purely imaginary.

In the last case (4) we consider only $\eta = 1$; the choice $\eta = -1$ here means that $\vec{a} = -\vec{a}$ and as a consequence we have $J = 0$, i.e. like in case (2) no N-wave equations are possible for $\eta = -1$. One easily finds that now the positive roots Δ_+ split into two subsets $\Delta_+ = \Delta_+^0 \cup \Delta_+^1$ such that $(\vec{c}, \alpha)$ is even for all $\alpha \in \Delta_+^0$ and odd for all $\alpha \in \Delta_+^1$. Obviously, if $\alpha \in \Delta_+^1$ then $s_\alpha = -1$ and the fields q_α, p_α must vanish due to (3.7*d*). As a result the effect of the reduction is to restrict us to an N-wave system related to the subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ with root system $\Delta^0 = \Delta_+^0 \cup (-\Delta_+^0)$. Such reductions are beyond the scope of this paper.

3.2. Reductions with Weyl group elements

For the second type of $\mathbb{Z}_2$-reductions, the C_j are related to Weyl group elements w such that $w^2 = \mathbb{1}$. Generically w is a composition of several Weyl reflections $S_{\beta_1} S_{\beta_2} \ldots$ where the roots $\beta_1, \beta_2, \ldots$ are pair-wise orthogonal. Fixing up w we can split Δ_+ into a union of four subsets

$$\Delta_+ \equiv \Delta_+^\perp \cup \Delta_+^\parallel \cup \Delta_+^+ \cup \Delta_+^- \tag{3.12}$$

where

$$w(\alpha) \equiv \alpha' = \alpha \qquad \text{for all} \quad \alpha \in \Delta_+^\perp \tag{3.13a}$$

$$w(\alpha) \equiv \alpha' = -\alpha \qquad \text{for all} \quad \alpha \in \Delta_+^\parallel \tag{3.13b}$$

$$w(\alpha) \equiv \alpha' > 0 \qquad \text{for all} \quad \alpha \in \Delta_+^+ \text{ and } \alpha \neq \alpha' \tag{3.13c}$$

$$w(\alpha) \equiv \alpha' < 0 \qquad \text{for all} \quad \alpha \in \Delta_+^- \text{ and } \alpha \neq -\alpha'. \tag{3.13d}$$

Depending on the choice of w one or more of these subsets may be empty. From now on we denote by $'$ the action of w on the corresponding root: $w(\alpha) = \alpha'$ and $w(\alpha') = \alpha$.

The subsets Δ_+^+ and Δ_+^- always contain even number of roots. Indeed, if $\alpha \in \Delta_+^+$ then α' also belongs to Δ_+^+. Analogously, if $\alpha \in \Delta_+^-$ then $-\alpha'$ also belongs to Δ_+^-. As we shall see below the reduction relates the coefficients Q_α with $Q_{\alpha'}$ or $Q_{-\alpha'}$. Therefore we will introduce the subsets $\tilde{\Delta}_+^\pm \subset \Delta_+^\pm$ satisfying

$$\tilde{\Delta}_+^+ \cup w(\tilde{\Delta}_+^+) = \Delta_+^+ \tag{3.14}$$

$$\tilde{\Delta}_+^- \cup \big(-w(\tilde{\Delta}_+^-)\big) = \Delta_+^-. \tag{3.15}$$

In other words, out of each pair $\{\alpha, \alpha'\} \in \Delta_+^+$ (resp. $\{\alpha, -\alpha'\} \in \Delta_+^-$) only one element belongs to $\tilde{\Delta}_+^+$ (resp. $\tilde{\Delta}_+^-$). For definiteness below we choose the element whose height is lower, i.e. $\alpha \in \tilde{\Delta}_+^\pm$ if $\mathrm{ht}\,(\alpha) < \mathrm{ht}\,(\pm\alpha')$.

We will also make use of the sets

$$\Delta_+^0 = \Delta_+^\perp \cup \Delta_+^+ \qquad \Delta_+^1 = \Delta_+^\parallel \cup \Delta_+^- \tag{3.16}$$

which obviously satisfy $w(\Delta_+^0) = \Delta_+^0$ and $w(\Delta_+^1) = -\Delta_+^1$.

The reduction conditions corresponding to each of the four types are most easily written down in terms of Q_α, see (1.3). Indeed, we have

$$(1) \quad Q_{\alpha'} = -\eta n_{\alpha,\alpha'} Q_{-\alpha}^* \qquad \vec{a} = \eta w_1(\vec{a}^*) \tag{3.17a}$$

$$(2) \quad Q_{\alpha'} = -\eta n_{\alpha,\alpha'} Q_{-\alpha} \qquad \vec{a} = -\eta w_2(\vec{a}) \tag{3.17b}$$

$$(3) \quad Q_{\alpha'} = \eta n_{\alpha,\alpha'} Q_{\alpha}^* \qquad \vec{a} = -\eta w_3(\vec{a}^*) \tag{3.17c}$$

$$(4) \quad Q_{\alpha'} = \eta n_{\alpha,\alpha'} Q_{\alpha} \qquad \vec{a} = \eta w_4(\vec{a}). \tag{3.17d}$$

Table 1. The set of independent fields for the reductions with Cartan subgroup elements.

Type	Complex	Real	Redundant
(1)	$q_\alpha, \quad \alpha \in \Delta_+$	—	—
(2)	$q_\alpha, \quad \alpha \in \Delta_+$	—	—
(3)	—	$q_\alpha, p_\alpha \quad \alpha \in \Delta_+$	—
(4)	$q_\alpha, p_\alpha \quad \alpha \in \Delta_+$ with $s_\alpha = 1$	—	$q_\alpha, p_\alpha \quad \alpha \in \Delta_+$ with $s_\alpha = -1$

Table 2. The set of independent fields for the reductions with Weyl subgroup elements. By $\Delta_{+,\varepsilon}^\perp$ and $\Delta_{+,\varepsilon}^\parallel$, $\varepsilon = \pm 1$ we denote the subsets of roots $\alpha \in \Delta_+^\perp$ (resp., $\alpha \in \Delta_+^\parallel$) for which $n_{\alpha\alpha'} = \varepsilon$.

Type	Complex	Real	Redundant
(1)	$q_\alpha, \quad \alpha \in \Delta_+^\perp \cup \tilde{\Delta}_+^+$ $q_\alpha, p_\alpha, \quad \alpha \in \tilde{\Delta}_+^-$	$q_\alpha, p_\alpha, \quad \alpha \in \Delta_+^\parallel$	—
(2)	$q_\alpha, \quad \alpha \in \Delta_+^\perp \cup \Delta_+^+$ $q_\alpha, p_\alpha, \quad \alpha \in \tilde{\Delta}_+^- \cup \Delta_{+,1}^\parallel$	—	$q_\alpha, p_\alpha, \quad \alpha \in \Delta_{+,-1}^\parallel$
(3)	$q_\alpha, \quad \alpha \in \Delta_+^\parallel \cup \Delta_+^-$ $q_\alpha, p_\alpha, \quad \alpha \in \tilde{\Delta}_+^+$	$q_\alpha, p_\alpha, \quad \alpha \in \Delta_+^\perp$	—
(4)	$q_\alpha, \quad \alpha \in \Delta_+^\parallel \cup \Delta_+^-$ $q_\alpha, p_\alpha, \quad \alpha \in \tilde{\Delta}_+^+ \cup \Delta_{+,1}^\perp$	—	$q_\alpha, p_\alpha, \quad \alpha \in \Delta_{+,-1}^\perp$

We will also describe the effect of the reduction on H and $\Omega^{(0)}$. Using the notation defined in equations (1.7)–(1.9) we can write

$$H = \sum_{\alpha \in \tilde{\Delta}_+^+} (H(\alpha) + H(\alpha')) + \sum_{\alpha \in \tilde{\Delta}_+^-} (H(\alpha) + H(-\alpha')) + \sum_{\alpha \in \Delta_+^\perp \cup \Delta_+^\parallel} H(\alpha) \tag{3.18}$$

$$\Omega^{(0)} = \sum_{\alpha \in \tilde{\Delta}_+^+} \left(\Omega(\alpha) + \Omega(\alpha')\right) + \sum_{\alpha \in \tilde{\Delta}_+^-} (\Omega(\alpha) + \Omega(-\alpha')) + \sum_{\alpha \in \Delta_+^\perp \cup \Delta_+^\parallel} \Omega(\alpha) \tag{3.19}$$

$$H(\alpha) = H_0(\alpha) + H_1(\alpha). \tag{3.20}$$

For the reductions (2.23) the restrictions on the potential matrix $Q(x, t)$ read as follows:

$$(1) \quad \begin{array}{lll} q_\alpha^* = -\eta n_{\alpha,\alpha'} p_{\alpha'} & p_\alpha^* = -\eta n_{\alpha,\alpha'} q_{\alpha'} & \text{for} \quad \alpha, \alpha' \in \Delta_+^0 \\ q_{-\alpha'}^* = -\eta n_{\alpha,\alpha'} q_\alpha & p_{-\alpha'}^* = -\eta n_{\alpha,\alpha'} p_\alpha & \text{for} \quad \alpha, -\alpha' \in \Delta_+^1 \end{array} \tag{3.21a}$$

$$(2) \quad \begin{array}{lll} q_\alpha = n_{\alpha,\alpha'} p_{\alpha'} & p_\alpha = n_{\alpha,\alpha'} q_{\alpha'} & \text{for} \quad \alpha, \alpha' \in \Delta_+^0 \quad \eta = -1 \\ q_\alpha = n_{\alpha,\alpha'} q_{-\alpha'} & p_\alpha = n_{\alpha,\alpha'} p_{-\alpha'} & \text{for} \quad \alpha, -\alpha' \in \Delta_+^1 \end{array} \tag{3.21b}$$

$$(3) \quad \begin{array}{lll} q_{\alpha'}^* = \eta n_{\alpha,\alpha'} q_\alpha & p_{\alpha'}^* = \eta n_{\alpha,\alpha'} p_\alpha & \text{for} \quad \alpha, \alpha' \in \Delta_+^0 \\ q_\alpha^* = \eta n_{\alpha,\alpha'} p_{-\alpha'} & p_\alpha^* = \eta n_{\alpha,\alpha'} q_{-\alpha'} & \text{for} \quad \alpha, -\alpha' \in \Delta_+^1 \end{array} \tag{3.21c}$$

$$(4) \quad \begin{array}{lll} q_\alpha = n_{\alpha,\alpha'} q_{\alpha'} & p_\alpha = n_{\alpha,\alpha'} p_{\alpha'} & \text{for} \quad \alpha, \alpha' \in \Delta_+^0 \quad \eta = 1 \\ p_\alpha = n_{\alpha,\alpha'} q_{-\alpha'} & q_\alpha = n_{\alpha,-\alpha'} p_{-\alpha'} & \text{for} \quad \alpha \in \Delta_+^1. \end{array} \tag{3.21d}$$

The set of independent fields for each of these reductions are collected in tables 1 and 2.

For the reductions of types (1) and (3), while the first two sets of variables are complex-valued, the fields related to the roots $\alpha \in \Delta_+^\parallel$ for (1) and the fields related to roots $\alpha \in \Delta_+^\perp$ for (3) should be either real or purely imaginary due to (3.21*a*) and (3.21*c*) respectively. Below, for the sake of brevity, we call them 'real'. In other words, after the reduction we get an N-wave system with $2|\Delta_+^\parallel|$ 'real' fields and $|\Delta_+^\perp| + |\Delta_+^+| + |\Delta_+^-|$ complex fields for the first reduction in (2.23) and for the third one we have $2|\Delta_+^\perp|$ real and $|\Delta_+^\parallel| + |\Delta_+^+| + |\Delta_+^-|$ complex functions.

The reduction conditions on $H(\alpha)$ and $\Omega^{(0)}(\alpha)$ read

$$H_{0,R}(\alpha) = -\mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{b},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q^*_{\alpha',x} - Q_{\alpha,x} Q^*_{\alpha'} \right) \tag{3.22}$$

$$\Omega^{(0)}_R(\alpha) = -\mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{a},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta Q_\alpha \wedge \delta Q^*_{\alpha'} \tag{3.23}$$

$$\left\{ \begin{matrix} \alpha \\ \beta,\gamma \end{matrix} \right\}_R = \left\{ \begin{matrix} \alpha' \\ \beta',\gamma' \end{matrix} \right\}^*_R = c_0 n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q^*_{\beta'} Q^*_{\gamma'} + \eta Q^*_{\alpha'} Q_\beta Q_\gamma \right) \tag{3.24}$$

for the first reduction in (2.23);

$$H_{0,R}(\alpha) = -\mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{b},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q_{\alpha',x} - Q_{\alpha,x} Q_{\alpha'} \right) \tag{3.25}$$

$$\Omega^{(0)}_R(\alpha) = -\mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{a},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta Q_\alpha \wedge \delta Q_{\alpha'} \tag{3.26}$$

$$\left\{ \begin{matrix} \alpha \\ \beta,\gamma \end{matrix} \right\}_R = \eta \left\{ \begin{matrix} \alpha' \\ \beta',\gamma' \end{matrix} \right\}_R = c_0 n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q_{\beta'} Q_{\gamma'} + \eta Q_{\alpha'} Q_\beta Q_\gamma \right) \tag{3.27}$$

for the second reduction in (2.23);

$$H_{0,R}(\alpha) = \mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{b},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q^*_{-\alpha',x} - Q_{\alpha,x} Q^*_{-\alpha'} \right) \tag{3.28}$$

$$\Omega^{(0)}_R(\alpha) = \mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{a},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta Q_\alpha \wedge \delta Q^*_{-\alpha'} \tag{3.29}$$

$$\left\{ \begin{matrix} \alpha \\ \beta,\gamma \end{matrix} \right\}_R = \left\{ \begin{matrix} -\alpha' \\ -\beta',-\gamma' \end{matrix} \right\}^*_R = c_0 n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q^*_{-\beta'} Q^*_{-\gamma'} - \eta Q^*_{-\alpha'} Q_\beta Q_\gamma \right) \tag{3.30}$$

for the third reduction in (2.23) and

$$H_{0,R}(\alpha) = \mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{b},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q_{-\alpha',x} - Q_{\alpha,x} Q_{-\alpha'} \right) \tag{3.31}$$

$$\Omega^{(0)}_R(\alpha) = \mathrm{i}\eta c_0 n_{\alpha,\alpha'} \frac{(\vec{a},\alpha)}{(\alpha,\alpha)} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta Q_\alpha \wedge \delta Q_{-\alpha'} \tag{3.32}$$

$$\left\{ \begin{matrix} \alpha \\ \beta,\gamma \end{matrix} \right\}_R = \eta \left\{ \begin{matrix} -\alpha' \\ -\beta',-\gamma' \end{matrix} \right\}_R = c_0 n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x \left(Q_\alpha Q_{-\beta'} Q_{-\gamma'} - \eta Q_{-\alpha'} Q_\beta Q_\gamma \right) \tag{3.33}$$

for the last reduction.

The general properties of $H(\alpha)$ and $\Omega^{(0)}(\alpha)$ are as follows:

$$(1)\quad H_0(\alpha) = H^*_0(\alpha') \qquad \Omega^{(0)}(\alpha) = (\Omega^{(0)}(\alpha'))^* \qquad \mathcal{H}_{\mathrm{I}}(\alpha) = \mathcal{H}^*_{\mathrm{I}}(\alpha') \tag{3.34}$$

$$(2)\quad H_0(\alpha) = \eta H_0(\alpha') \qquad \Omega^{(0)}(\alpha) = \eta(\Omega^{(0)}(\alpha')) \qquad \mathcal{H}_{\mathrm{I}}(\alpha) = \eta\mathcal{H}_{\mathrm{I}}(\alpha') \tag{3.35}$$

$$(3)\quad H_0(\alpha) = H^*_0(\alpha') \qquad \Omega^{(0)}(\alpha) = (\Omega^{(0)}(\alpha'))^* \qquad \mathcal{H}_{\mathrm{I}}(\alpha) = \mathcal{H}^*_{\mathrm{I}}(\alpha') \tag{3.36}$$

$$(4)\quad H_0(\alpha) = \eta H_0(\alpha') \qquad \Omega^{(0)}(\alpha) = \eta(\Omega^{(0)}(\alpha')) \qquad \mathcal{H}_{\mathrm{I}}(\alpha) = \eta\mathcal{H}_{\mathrm{I}}(\alpha'). \tag{3.37}$$

Obviously, if $\alpha \in \Delta_+^\perp$, i.e. $\alpha' = \alpha$, then the expressions in the right-hand side of (3.22) and (3.23) coincide with the ones in (3.9) and (3.11). However, if $\alpha' \neq \alpha > 0$ (i.e., if $\alpha \in \Delta_+^+$) the field variable q_α will be dynamically conjugated not to q^*_α but to $q^*_{\alpha'}$. The same holds true for the second reduction in (2.23): if $\alpha \in \Delta^\perp$ the expressions in the left-hand side of (3.25) and (3.26) coincide with the general ones. This fact makes these reduced N-wave systems substantially different from the ones described in section 3.1.

In appendix A we list the sets $\mathcal{M}_\alpha$ of all pairs of roots β,γ such that $\beta + \gamma = \alpha$ and the coefficients ω_{jk} (1.9). In appendix B we give explicit expressions for some of the specific reduced interaction Hamiltonian terms.

3.3. Inequivalent embeddings of $\mathbb{Z}_2$ in $W_{\mathfrak{g}}$

The reduction group G_R may be embedded in the Weyl group $W(\mathfrak{g})$ of the simple Lie algebra in a number of ways. Therefore it will be important to have a criterion to distinguish the nonequivalent reductions. As any other finite group, $W(\mathfrak{g})$ can be split into equivalence classes. So one may expect that reductions with elements from the same equivalence class would lead to equivalent reductions; namely the two systems of N-wave equations will be related by a change of variables.

In what follows we will describe the equivalence classes of the Weyl groups $W(\boldsymbol{B}_2)$, $W(\boldsymbol{G}_2)$ and $W(\boldsymbol{B}_3)$; note that $W(\boldsymbol{B}_l) \simeq W(\boldsymbol{C}_l)$. This is due to two facts: (1) the system of positive roots for $\boldsymbol{B}_r$ is $\Delta^+_{\boldsymbol{B}_r} \equiv \{e_i \pm e_j, e_i\}$, $i < j$ while the one for $\boldsymbol{C}_r$ series is $\Delta^+_{\boldsymbol{C}_r} \equiv \{e_i \pm e_j, 2e_i\}$, $i < j$; and (2) the reflection S_{e_j} with respect to the root e_j coincide with S_{2e_j}—the one with respect to the root $2e_j$. In the tables below we provide for each equivalence class: (i) the cyclic group generated by each of the automorphisms in the class; (ii) the number of elements in each class and (iii) a representative element in it.

Remark 1. For $\boldsymbol{B}_r$ and $\boldsymbol{C}_r$ series and for $\boldsymbol{G}_2$ the inner automorphism w_0 which maps the highest-weight vectors into the lowest-weight vectors of the algebra acts on the Cartan–Weyl basis as follows:

$$w_0(E_\alpha) = n_\alpha E_{-\alpha} \qquad w_0(H_k) = -H_k \qquad \alpha \in \Delta_+ \quad n_\alpha = \pm 1. \tag{3.38}$$

Let us list the genetic codes of the Weyl groups for these Lie algebras:

$$W(\boldsymbol{A}_2) \simeq \mathbb{D}_3 \qquad S^2_{e_1-e_2} = S^2_{e_2-e_3} = 1\!\!1 \qquad (S_{e_1-e_2}S_{e_2-e_3})^3 = 1\!\!1 \tag{3.39}$$

$$W(\boldsymbol{B}_2) \simeq \mathbb{D}_4 \qquad S^2_{e_1-e_2} = S^2_{e_2} = 1\!\!1 \qquad (S_{e_1-e_2}S_{e_2})^4 = 1\!\!1 \tag{3.40}$$

$$W(\boldsymbol{G}_2) \simeq \mathbb{D}_6 \qquad S^2_{e_1-e_2} = S^2_{e_2} = 1\!\!1 \qquad (S_{e_1-e_2}S_{e_2})^6 = 1\!\!1 \tag{3.41}$$

$$\begin{aligned} W(\boldsymbol{A}_3) \simeq \mathcal{S}_4 \qquad & S^2_{e_1-e_2} = S^2_{e_2-e_3} = S^2_{e_3-e_4} = 1\!\!1 \qquad (S_{e_1-e_2}S_{e_2-e_3})^3 = 1\!\!1 \\ & (S_{e_1-e_2}S_{e_2-e_3}S_{e_3-e_4})^4 = 1\!\!1 \end{aligned} \tag{3.42}$$

$$\begin{aligned} W(\boldsymbol{B}_3) \qquad & S^2_{e_1-e_2} = S^2_{e_2-e_3} = S^2_{e_3} = 1\!\!1 \qquad (S_{e_1-e_2}S_{e_2-e_3})^3 = 1\!\!1 \\ & (S_{e_2-e_3}S_{e_3})^4 = 1\!\!1 \qquad (S_{e_1-e_2}S_{e_2-e_3}S_{e_3})^6 = 1\!\!1 \end{aligned} \tag{3.43}$$

where $\mathcal{S}_4$ is the group of permutation of four elements.

Their equivalence classes are listed in the tables below, where in the first line we denote the order of each of elements in the class, in the second line we list the number of elements in each class, and in the third line give a representative element.

$\boldsymbol{A}_2$

$1\!\!1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
1	3	2
$1\!\!1$	$S_{e_1-e_2}$	$S_{e_1-e_2}S_{e_2-e_3}$

$\boldsymbol{B}_2$

$1\!\!1$	$-1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4$
1	1	2	2	2
$1\!\!1$	w_0	$S_{e_1-e_2}$	S_{e_1}	$S_{e_1-e_2}S_{e_2}$

$\boldsymbol{G}_2$

$1\!\!1$	$-1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_3$	$\mathbb{Z}_6$
1	1	3	3	2	2
$1\!\!1$	w_0	S_{α_1}	S_{α_2}	$(S_{\alpha_1}S_{\alpha_2})^2$	$S_{\alpha_1}S_{\alpha_2}$

$\boldsymbol{A}_3$

$1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_3$	$\mathbb{Z}_4$
1	6	3	8	6
$1\!\!1$	$S_{e_1-e_2}$	$S_{e_1-e_2}S_{e_3-e_4}$	$S_{e_1-e_2}S_{e_2-e_3}$	$S_{e_1-e_2}S_{e_2-e_3}S_{e_3-e_4}$

$\boldsymbol{B}_3$

$\mathbb{1}$ 1 $\mathbb{1}$	$-\mathbb{1}$ 1 w_0	$\mathbb{Z}_2^{(1)}$ 6 $S_{e_1-e_2}$	$\mathbb{Z}_2^{(2)}$ 3 S_{e_3}	$\mathbb{Z}_2^{(3)}$ 6 $S_{e_1-e_2}S_{e_3}$
$\mathbb{Z}_2^{(4)}$ 3 $S_{e_1}S_{e_2}$	$\mathbb{Z}_3$ 8 $S_{e_1-e_2}S_{e_2-e_3}$	$\mathbb{Z}_4^{(1)}$ 6 $S_{e_1}S_{e_1-e_2}$	$\mathbb{Z}_4^{(2)}$ 6 $S_{e_1}S_{e_3}S_{e_1-e_2}$	$\mathbb{Z}_6$ 8 $S_{e_1-e_2}S_{e_2-e_3}S_{e_3}$

We leave more detailed explanations of the general theory of finite groups to other papers and turn now to the examples.

Remark 2. In all examples below we apply the reductions to L-operators of generic form. This means that the unreduced J is a generic element of $\mathfrak{h}$ and therefore $(\vec{a}, \alpha) \neq 0$. In fact we have used above the vector $\vec{a}$ for fixing up the order in the root system of $\mathfrak{g}$. The potential Q is also generic, i.e. depends on $|\Delta|$ complex-valued functions where $|\Delta|$ is the number of roots of $\mathfrak{g}$. However, the reduction imposed on J may lead to a qualitatively different situation in which the reduced $J_{\rm r}$ is not generic, i.e. there may exist a subset of roots Δ_0 such that $(\vec{a}_{\rm r}, \alpha) = 0$ for $\alpha \in \Delta_0$. Then, obviously, the potential $[J, Q]$ in L will depend only on $|\Delta| - |\Delta_0|$ complex-valued fields, the other fields being redundant, see tables 1 and 2.

In what follows, whenever such situations arise we will provide the subset Δ_0 or, equivalently, the list of redundant functions in Q. Obviously, both the corresponding N-wave equation and its Hamiltonian structures will depend only on the fields labelled by the roots α such that $(\vec{a}_{\rm r}, \alpha) \neq 0$, see [19].

Remark 3. Several of the $\mathbb{Z}_2$-reductions below contain automorphisms which map J to $-J$. Then it is only natural that both the canonical symplectic form $\Omega^{(0)}$ and the Hamiltonian $H^{(0)}$ vanish identically. In these cases we will write down the corresponding N-wave systems of equations; their Hamiltonian formulation is discussed in section 5.

Remark 4. Under some of the reductions the corresponding equation (1.1) becomes linear and trivial. This happens when the Cartan subalgebra elements invariant under the reduction form a one-dimensional subspace in $\mathfrak{h}$ and therefore $J_{\rm r} \propto I_{\rm r}$. For obvious reasons we have omitted these examples.

4. Description of the $\mathbb{Z}_2$ reductions

Remark 5. In what follows we will skip the leading zeros in the notation of the roots: for example, by $\{1\}$ and $\{11\}$ we mean $\{001\}$ and $\{011\}$ respectively for the $\boldsymbol{A}_3$, $\boldsymbol{B}_3$ and $\boldsymbol{C}_3$ algebras. For $\boldsymbol{A}_2$, $\boldsymbol{C}_2$ and $\boldsymbol{G}_2$ algebra by $\{1\}$ we mean $\{01\}$. We also drop all indices R in the triples $\left\{ \begin{array}{c} \alpha \\ \beta, \gamma \end{array} \right\}$.

4.1. $\mathfrak{g} \simeq \boldsymbol{A}_2 = sl(3)$

This algebra has three positive roots $\Delta^+ = \{10, 01, 11\}$ where $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$ and $jk = j\alpha_1 + k\alpha_2$. Then $Q(x, t)$ contains six functions and the set $\mathcal{M}$ contains only one triple $\mathcal{M} \equiv \{[11, 01, 10]\}$.

Example 1. $C_1 = \mathbb{1}$. $U^T(-\lambda) + U(\lambda) = 0$. This reduction does not restrict the Cartan elements. We have

$$p_\alpha = q_\alpha \qquad \alpha \in \Delta_+ \tag{4.1}$$

and we obtain the next 3-wave system:

$$\begin{aligned} &\mathrm{i}(a_1 - a_2)q_{10,t} - \mathrm{i}(b_1 - b_2)q_{10,x} - \kappa q_1 q_{11} = 0 \\ &\mathrm{i}(a_2 - a_3)q_{1,t} - \mathrm{i}(b_2 - b_3)q_{1,x} - \kappa q_{10} q_{11} = 0 \\ &\mathrm{i}(a_1 - a_3)q_{11,t} - \mathrm{i}(b_1 - b_3)q_{11,x} + \kappa q_{10} q_1 = 0 \end{aligned} \tag{4.2}$$

with $\kappa = a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_3b_2 - a_1b_3$. Due to the reduction conditions for the elements of the potential matrix the Hamiltonian vanishes, see remark 3.

Example 2. $C_2 = \mathbb{1}$. $U^*(\eta\lambda^*) + U(\lambda) = 0$. Therefore

$$a_i^* = -\eta a_i \qquad b_i^* = -\eta b_i \qquad p_\alpha^* = \eta p_\alpha \qquad q_\alpha^* = \eta q_\alpha \tag{4.3}$$

and we obtain 6 'real' fields and the 6-wave system with the following Hamiltonian:

$$H^{(0)} = H_0(10) + H_0(01) + H_0(11) + \kappa H(11, 1, 10). \tag{4.4}$$

Here again $\kappa = a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_3b_2 - a_1b_3$. The case $\eta = 1$ leads to the noncompact real form $sl(3, \mathbb{R})$ for the $\boldsymbol{A}_2$-algebra.

Example 3. $C_3 = S_{e_1-e_3}$. $C_3(U^*(\eta\lambda^*)) + U(\lambda) = 0$. We have

$$\begin{aligned} &a_3 = \eta a_1^* \qquad a_2^* = \eta a_2 \qquad b_3 = \eta b_1^* \qquad b_2^* = \eta b_2 \\ &p_{10}^* = -\eta q_1^* \qquad p_1 = -\eta q_{10}^* \qquad p_{11} = -\eta q_{11}^* \end{aligned} \tag{4.5}$$

and we obtain the 3-wave system with the Hamiltonian

$$H^{(0)} = H_{0*}(10) + H_{0*}(01) + H_{0*}(11) - \kappa H_*(11, 1, 10) \tag{4.6}$$

where

$$H_*(\alpha, \beta, \gamma) = \frac{1}{\sqrt{\eta}} \int_{-\infty}^{\infty} \mathrm{d}x \, (q_\alpha q_\beta^* q_\gamma^* + \eta q_\alpha^* q_\beta q_\gamma) \tag{4.7}$$

and again $\kappa = a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_3b_2 - a_1b_3$.

Example 4. $C_4 = \mathbb{1}$. $U^\dagger(\eta\lambda^*) - U(\lambda) = 0$. Therefore

$$a_i^* = \eta a_i \qquad b_i^* = \eta b_i \qquad p_\alpha = -\eta q_\alpha^* \tag{4.8}$$

and we obtain the 3-wave system with the following Hamiltonian:

$$H^{(0)} = H_{0*}(11) + H_{0*}(10) + H_{0*}(01) + \kappa H_*(11, 1, 10). \tag{4.9}$$

Here again $\kappa = a_1b_2+a_2b_3+a_3b_1-a_2b_1-a_3b_2-a_1b_3$ and $H_*(11, 01, 10)$ is defined by (4.7). The case $\eta = 1$ extracts the compact real form $su(3)$ for the $\boldsymbol{A}_2$-algebra.

Example 5. $C_5 = S_{e_1-e_3}$. $C_5(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$. We obtain

$$\begin{aligned} &a_3 = \eta a_1^* \qquad a_2^* = \eta a_2 \qquad b_3 = \eta b_1^* \qquad b_2^* = \eta b_2 \\ &q_1 = \eta q_{10}^* \qquad q_{11}^* = -\eta q_{11} \qquad p_1 = \eta p_{10}^* \qquad p_{11}^* = -\eta p_{11} \end{aligned} \tag{4.10}$$

and we obtain the 4-wave (2 real and 2 complex) system with the Hamiltonian

$$H^{(0)} = H_0(11) + 2\mathrm{Re}\,(H_0(01) + H_0(10)) - \frac{1}{\sqrt{-\eta}} \int_{-\infty}^{\infty} \mathrm{d}x \, (p_{11}|q_{10}|^2 - q_{11}|p_{10}|^2). \tag{4.11}$$

Here again $\kappa = a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_3b_2 - a_1b_3$ is real.

Example 6. $C_6 = \Sigma = \mathrm{diag}\,(s_1, s_2, s_3)$. $\Sigma U^\dagger(\eta\lambda^*)\Sigma^{-1} - U(\lambda) = 0$ and $s_i = \pm 1$. This reduction restricts the Cartan elements to be real (purely imaginary) for $\eta = 1$ ($\eta = -1$) and

$$p_{10} = -\eta \frac{s_2}{s_1} q_{10}^* \qquad p_1 = -\eta \frac{s_3}{s_2} q_1^* \qquad p_{11} = -\eta \frac{s_3}{s_1} q_{11}^*. \tag{4.12}$$

Thus we get the 3-wave system with the Hamiltonian

$$H^{(0)} = \frac{s_2}{s_1} H_{0*}(10) + \frac{s_3}{s_2} H_{0*}(01) + \frac{s_3}{s_1} H_{0*}(11) + \kappa H_*(11, 1, 10). \tag{4.13}$$

Here again $\kappa = a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_3b_2 - a_1b_3$ and $H_*(i, j, k)$ is defined by (4.7). The choice $\eta = 1$, $s_1 = s_2 = s_3$ reproduces the result of example 4 while the choice $\eta = 1$, $s_1 = -s_2 = -s_3$ extracts the noncompact real form $su(2, 1)$ of the $\boldsymbol{A}_2$-algebra.

4.2. $\mathfrak{g} \simeq \boldsymbol{C}_2 = sp(4)$

This algebra has four positive roots $\Delta^+ = \{10, 01, 11, 21\}$ where $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_2$ and $jk = j\alpha_1 + k\alpha_2$. Then $Q(x, t)$ contains eight functions. The set $\mathcal{M}$ consists of two elements: $\mathcal{M} = \{[21, 11, 10], [11, 01, 10]\}$.

Example 7. $C_7 = \mathbb{I}$. $U^*(\eta\lambda^*) + U(\lambda) = 0$, $\eta = \pm 1$. Then all functions q_α, p_α become real and the Cartan elements become purely imaginary for $\eta = 1$ and vice versa for $\eta = -1$, i.e.

$$a_i = -\eta a_i^* \qquad b_i = -\eta b_i^* \qquad i = 1, 2 \qquad q_\alpha = \eta q_\alpha^* \qquad p_\alpha = \eta p_\alpha^* \qquad \alpha \in \Delta_+. \tag{4.14}$$

Thus we obtain 8 'real' fields and the 8-wave system with the Hamiltonian

$$H^{(0)} = H_0(10) + H_0(1) + H_0(11) + H_0(21) + 2\kappa(H(21, 11, 10) - H(11, 1, 10)) \tag{4.15}$$

with $\kappa = (a_1b_2 - a_2b_1)$ which is related to the noncompact real form $sp(4, \mathbb{R})$ of the $\boldsymbol{C}_2$-algebra.

Example 8. $C_8 = w_0$. $C_8(U^*(\eta\lambda^*)) + U(\lambda) = 0$ and $\eta = \pm 1$. Then

$$a_1^* = \eta a_1 \qquad a_2^* = \eta a_2 \qquad b_1^* = \eta b_1 \qquad b_2^* = \eta b_2 \qquad p_\alpha = -\eta q_\alpha^* \tag{4.16}$$

which leads to the general 4-wave system on the compact real form $sp(4, 0)$ $(\eta = 1)$ of $\boldsymbol{C}_2$ algebra with the Hamiltonian

$$\begin{aligned} H^{(0)} = {} & H_{0*}(10) + H_{0*}(1) + H_{0*}(11) + H_{0*}(21) \\ & + \frac{2\kappa}{\sqrt{\eta}} \int_{-\infty}^{\infty} \mathrm{d}x\, [q_{11}q_{10}^*q_1^* + q_{21}q_{11}^*q_{10}^* + \eta(q_{11}^*q_{10}q_1 + q_{21}^*q_{11}q_{10})] \end{aligned} \tag{4.17}$$

and $\kappa = a_1b_2 - a_2b_1$.

Example 9. After a reduction of Hermitian type $K^{-1}U^\dagger(\eta\lambda^*)K - U(\lambda) = 0$, where $K = \mathrm{diag}\,(s_1, s_2, 1/s_2, 1/s_1)$ and $\eta = \pm 1$ we obtain

$$\begin{aligned} & p_{10} = -\eta s_1/s_2 q_{10}^* \qquad p_1 = -\eta s_2^2 q_1^*, \quad p_{11} = -\eta s_1 s_2 q_{11}^* \qquad p_{21} = -\eta s_1^2 q_{21}^* \\ & a_i = \eta a_i^* \qquad b_i = \eta b_i^* \end{aligned} \tag{4.18}$$

and the next 4-wave system

$$\begin{aligned} & \mathrm{i}(a_1 - a_2)q_{10;t} - \mathrm{i}(b_1 - b_2)q_{10;x} + 2\eta\kappa(s_2^2 q_{11}q_1^* - s_1s_2q_{21}q_{11}^*) = 0 \\ & \mathrm{i}a_2q_{1;t} - \mathrm{i}b_2q_{1;x} + 2\eta\kappa(s_1/s_2)q_{11}q_{10}^* = 0 \\ & \mathrm{i}a_1q_{21;t} - \mathrm{i}b_4q_{21;x} + 2\kappa q_{11}q_{10} = 0 \\ & \mathrm{i}(a_1 + a_2)q_{11;t} - \mathrm{i}(b_1 + b_2)q_{11;x} - 2\kappa(q_{10}q_1 + \eta(s_1/s_2)q_{21}q_{10}^*) = 0 \end{aligned} \tag{4.19}$$

where $\kappa = a_1b_2 - a_2b_1$. It is described by the following Hamiltonian:

$$\begin{aligned} H^{(0)} = {} & \frac{s_1}{s_2}H_{0*}(10) + s_2^2H_{0*}(1) + s_1s_2H_{0*}(11) + s_1^2H_{0*}(21) \\ & + \frac{2\kappa}{\sqrt{\eta}} \int_{-\infty}^{\infty} \mathrm{d}x\, (s_1s_2(q_{11}q_1^*q_{10}^* + \eta q_{11}^*q_1q_{10}) - s_1^2(q_{21}q_{11}^*q_{10}^* + \eta q_{21}^*q_{11}q_{10})). \end{aligned} \tag{4.20}$$

In the case $\eta = -1$ if we identify $q_{10} = Q$, $q_{11} = E_p$, $q_{21} = E_a$ and $q_1 = E_s$, where Q is the normalized effective polarization of the medium and E_p, E_s and E_a are the normalized pump, Stokes and anti-Stokes wave amplitudes respectively, then we obtain the system of equations generalizing the one studied in [23] which describes Stokes–anti-Stokes wave generation. This approach allowed us to derive a new Lax pair for (4.19). A particular case of (4.19) with $s_1 = s_2 = \pm 1$ and $\eta = \pm 1$ is equivalent to the 4-wave interaction, see [4], and is related to the compact real form $sp(4, 0)$ of $\boldsymbol{C}_2$. For $s_1 = -s_2 = \pm 1$ and $\eta = 1$ the reduced system is related to the noncompact real form $sp(2, 2)$ of the $\boldsymbol{C}_2$-algebra.

Example 10. $C_{10} = S_{e_1-e_2}$. $C_{10}(U^*(\eta\lambda^*)) + U(\lambda) = 0$ and $\eta = \pm 1$. This reduction gives the following restrictions:

$$\begin{aligned} &a_2 = -\eta a_1^* \qquad b_2 = -\eta b_1^* \\ &p_{10} = \eta q_{10}^* \qquad q_{11}^* = \eta q_{11} \qquad q_{21} = -\eta q_1^* \qquad p_{11}^* = \eta p_{11} \qquad p_{21} = -\eta p_1^*. \end{aligned} \tag{4.21}$$

Then we obtain the 5-wave (2 real and 3 complex) system which is described by the Hamiltonian

$$\begin{aligned} H^{(0)} = {} & H_{0*}(10) + H_0(11) + 2\mathrm{Re}\, H_0(1) \\ & + \frac{2\kappa}{\sqrt{\eta}} \int_{-\infty}^{\infty} \mathrm{d}x\, [q_{11}(q_{10}^* p_1 - q_{10} p_1^*) + \eta p_{11}(q_{10}^* q_1^* - q_{10} q_1)] \end{aligned} \tag{4.22}$$

with $\kappa = a_1 b_1^* - a_1^* b_1$.

Example 11. $C_{11} = S_{2e_2}$. $C_{11}(U^*(\eta\lambda^*)) + U(\lambda) = 0$ and $\eta = \pm 1$. Then we have

$$\begin{aligned} &a_1^* = -\eta a_1 \qquad a_2^* = \eta a_2 \qquad b_1^* = -\eta b_1 \qquad b_2^* = \eta b_2 \\ &q_{11} = -\mathrm{i}\eta q_{10}^* \qquad p_{11} = \mathrm{i}\eta p_{10}^* \qquad q_{21}^* = -\eta q_{21} \\ &p_{21}^* = -\eta p_{21} \qquad p_1 = -\eta q_1^* \end{aligned} \tag{4.23}$$

which leads again to the 5-wave (2 real and 3 complex) system with the Hamiltonian

$$\begin{aligned} H^{(0)} = {} & 2\mathrm{Re}\, H_0(10) + H_{0*}(1) + H_0(21) \\ & + \frac{2\mathrm{i}\kappa}{\sqrt{\eta}} \int_{-\infty}^{\infty} \mathrm{d}x\, [p_{10} q_1^* q_{10}^* - \eta(p_{10}^* q_1 q_{10} + p_{21}|q_{10}|^2 + q_{21}|p_{10}|^2)] \end{aligned} \tag{4.24}$$

and $\kappa = a_1 b_2 - a_2 b_1$.

Example 12. $C_{12} = w_0$. $C_{12}(U(-\lambda)) - U(\lambda) = 0$. Here we get

$$p_{10} = q_{10} \qquad p_{11} = q_{11} \qquad p_1 = q_1 \qquad p_{21} = q_{21}. \tag{4.25}$$

Then we obtain the following 4-wave system, see remark 3:

$$\begin{aligned} &\mathrm{i}(a_1 - a_2) q_{10,t} - \mathrm{i}(b_1 - b_2) q_{10,x} - 2\kappa(q_{21} q_{11} + q_1 q_{11}) = 0 \\ &\mathrm{i}a_2 q_{1,t} - \mathrm{i}b_2 q_{1,x} - 2\kappa q_{10} q_{11} = 0 \\ &\mathrm{i}(a_1 + a_2) q_{11,t} - \mathrm{i}(b_1 + b_2) q_{11,x} + 2\kappa(q_{21} q_{10} - q_1 q_{10}) = 0 \\ &\mathrm{i}a_1 q_{21,t} - \mathrm{i}b_1 q_{21,x} + 2\kappa q_{10} q_{11} = 0 \end{aligned} \tag{4.26}$$

with $\kappa = a_1 b_2 - a_2 b_1$. Note that this reduction does not restrict the Cartan elements.

4.3. $\mathfrak{g} \simeq \boldsymbol{G}_2$

$\boldsymbol{G}_2$ has six positive roots $\Delta^+ = \{10, 01, 11, 21, 31, 32\}$ where again $km = k\alpha_1 + m\alpha_2$, $\alpha_1 = (e_1 - e_2 + 2e_3)/3$, $\alpha_2 = e_2 - e_3$ and the interaction Hamiltonian contains the set of triples of indices $\mathcal{M} \equiv \{[11, 1, 10], [21, 11, 10], [31, 21, 10], [32, 31, 1], [32, 21, 11]\}$.

Note that here if the Cartan elements are real then the N-wave equations after the reduction become trivial except one, case 13, see remark 4.

Example 13. $C_{13} = 1\!\!1$. $C_{13}(U^T(-\lambda)) + U(\lambda) = 0$. This does not restrict the Cartan elements and for the potential matrix gives

$$p_\alpha = q_\alpha \qquad \alpha \in \Delta_+ \tag{4.27}$$

and a 6–wave system, see remark 3,

$$\begin{aligned}
&\mathrm{i}(2a_1 - a_2)q_{10,t} - \mathrm{i}(2b_1 - b_2)q_{10,x} + \kappa(q_1q_{11} + 2q_{21}q_{11} + q_{31}q_{21}) = 0\\
&\mathrm{i}(3a_1 - a_2)q_{1,t} - \mathrm{i}(3b_1 - b_2)q_{1,x} - 3\kappa(q_{10}q_{11} + q_{32}q_{31}) = 0\\
&\mathrm{i}(a_1 - a_2)q_{11,t} - \mathrm{i}(b_1 - b_2)q_{11,x} + \kappa(q_1q_{10} - 2q_{21}q_{10} + q_{32}q_{21}) = 0\\
&\mathrm{i}a_1q_{21,t} - \mathrm{i}b_1q_{21,x} - \kappa(2q_{11}q_{10} - q_{31}q_{10} + q_{32}q_{11}) = 0\\
&\mathrm{i}(3a_1 - a_2)q_{31,t} - \mathrm{i}(3b_1 - b_2)q_{31,x} + 3\kappa(q_{32}q_1 - q_{21}q_{10}) = 0\\
&\mathrm{i}a_2q_{32,t} - \mathrm{i}b_2q_{32,x} + 3\kappa(q_{21}q_{11} - q_{31}q_1) = 0
\end{aligned} \tag{4.28}$$

with $\kappa = a_1b_2 - a_2b_1$. Due to the reduction conditions for the potential matrix (4.27) the terms $H(\alpha, \beta, \gamma)$ in (1.9) vanish.

Example 14. $C_{14} = w_0$. $C_{14}(U^*(\eta\lambda^*)) + U(\lambda) = 0$ and $\eta = \pm 1$. This gives

$$a_1^* = \eta a_1 \qquad a_2^* = \eta a_2 \qquad b_1^* = \eta b_1 \qquad b_2^* = \eta b_2 \qquad p_\alpha = -\eta q_\alpha^* \tag{4.29}$$

and a 6–wave system described by the Hamiltonian

$$\begin{aligned}
H^{(0)} = {} & 3(H_{0*}(10) + H_{0*}(11) + H_{0*}(21)) + H_{0*}(1) + H_{0*}(31) + H_{0*}(32)\\
&+3\kappa[H_*(32, 31, 1) + H_*(32, 21, 11) - H_*(31, 21, 10)\\
&-2H_*(21, 11, 10) + H_*(11, 10, 1)]
\end{aligned} \tag{4.30}$$

with $\kappa = a_1b_2 - a_2b_1$.

Example 15. $\Sigma^{-1}U^\dagger(\eta\lambda^*)\Sigma - U(\lambda) = 0$, $\eta = \pm 1$ where Σ belongs to the Cartan subgroup and equals $\Sigma = \mathrm{diag}\,(s_1s_2, s_1, s_2, 1, 1/s_2, 1/s_1, 1/(s_1s_2))$. Then all Cartan elements become real (purely imaginary) for $\eta = 1$ ($\eta = -1$) and

$$\begin{aligned}
&p_{10} = -\eta\frac{1}{s_2}q_{10}^* && p_1 = -\eta\frac{s_2}{s_1}q_1^* && p_{11} = -\eta\frac{1}{s_1}q_{11}^*\\
&p_{21} = -\eta\frac{1}{s_1s_2}q_{21}^* && p_{31} = -\eta\frac{1}{s_1s_2^2}q_{31}^* && p_{32} = -\eta\frac{1}{s_1^2s_2}q_{32}^*
\end{aligned} \tag{4.31}$$

which leads to a 6-wave system with Hamiltonian

$$\begin{aligned}
H^{(0)} = {} & 3\left(\frac{1}{s_2}H_{0*}(10) + \frac{1}{s_1}H_{0*}(11) + \frac{1}{s_1s_2}H_{0*}(21) + \frac{s_2}{s_1}H_{0*}(1) + \frac{1}{s_1s_2^2}H_{0*}(31)\right.\\
&\left.+\frac{1}{s_1^2s_2}H_{0*}(32)\right) + 3\kappa\left[\frac{1}{s_1^2s_2}H_*(32, 31, 1) + \frac{1}{s_1^2s_2}H_*(32, 21, 11)\right.\\
&\left.-\frac{1}{s_1s_2^2}H_*(31, 21, 10) - \frac{2}{s_1s_2}H_*(21, 11, 10) + \frac{1}{s_1}H_*(11, 10, 1)\right]
\end{aligned} \tag{4.32}$$

with $\kappa = a_1b_2 - a_2b_1$ and $H_*(\alpha, \beta, \gamma)$ is defined by (4.7). In the particular case $s_1 = s_2 = 1$ we obtain the result of example 14, $\eta = 1$.

Example 16. $C_{16} = S_{\alpha_1}$. $C_{16}(U^*(\eta\lambda^*)) + U(\lambda) = 0$ and $\eta = \pm 1$. Then

$$\begin{aligned}
&a_2 = a_1 - \eta a_1^* && b_2 = b_1 - \eta b_1^* && &&\\
&q_{31} = \eta q_1^* && p_{10} = \eta q_{10}^* && q_{21} = \eta q_{11}^* && q_{32}^* = \eta q_{32}\\
&p_{31} = \eta p_1^* && p_{21} = \eta p_{11}^* && p_{32}^* = \eta p_{32}. &&
\end{aligned} \tag{4.33}$$

So we obtain the 7-wave (2 real and 5 complex) system with the Hamiltonian

$$H^{(0)} = -H_{0*}(10) + 2\mathrm{Re}\, H_0(31) + 2\mathrm{Re}\, H_0(21) + H_0(32) + \left\{ \begin{matrix} 32 \\ 21, 11' \end{matrix} \right\} + \left\{ \begin{matrix} 32 \\ 31, 01' \end{matrix} \right\} + \left\{ \begin{matrix} 21 \\ 11', 10 \end{matrix} \right\} + 2\mathrm{Re} \left\{ \begin{matrix} 31 \\ 21, 10 \end{matrix} \right\}. \tag{4.34}$$

Example 17. $C_{17} = S_{\alpha_2}$. $C_{17}(U^*(\eta\lambda^*)) + U(\lambda) = 0$ and $\eta = \pm 1$. Then

$$\begin{array}{llll} a_1 = \frac{1}{3}(a_2 - \eta a_2^*) & b_1 = \frac{1}{3}(b_2 - \eta b_2^*) & & \\ q_{11} = -\eta q_{10}^* & p_1 = \eta q_1^* & q_{21}^* = -\eta q_{21} & q_{32} = \eta q_{31}^* \\ p_{11} = -\eta p_{10}^* & p_{21}^* = -\eta p_{21} & p_{32} = \eta p_{31}^*. & \end{array} \tag{4.35}$$

So we obtain the 7-wave (2 real and 5 complex) system which is described by the Hamiltonian

$$H^{(0)} = 2\mathrm{Re}\, H_0(11) - H_{0*}(1) + H_0(21) + 2\mathrm{Re}\, H_0(32) + 2\mathrm{Re} \left\{ \begin{matrix} 32 \\ 21, 11 \end{matrix} \right\} + \left\{ \begin{matrix} 32 \\ 31', 01 \end{matrix} \right\} + \left\{ \begin{matrix} 21 \\ 11, 10' \end{matrix} \right\} + \left\{ \begin{matrix} 11 \\ 01, 10' \end{matrix} \right\} \tag{4.36}$$

with $\kappa = a_2 b_2^* - a_2^* b_2$.

4.4. $\mathfrak{g} \simeq \boldsymbol{A}_3 = sl(4)$

This algebra has six positive roots: $\Delta_+ = \{100, 010, 001, 110, 011, 111\}$ where again $ijk = i\alpha_1 + j\alpha_2 + k\alpha_3$ and $\alpha_1 = e_1 - e_2$; $\alpha_2 = e_2 - e_3$; $\alpha_3 = e_3 - e_4$ are the simple roots of the $\boldsymbol{A}_3$-algebra. The set $\mathcal{M}$ consists of

$$\mathcal{M} = \{[111, 011, 100], [111, 110, 001], [011, 001, 010], [110, 010, 100]\}.$$

Example 18. $C_{18} = S_{e_1 - e_2}$. $C_{18}(U(\lambda)) - U(\lambda) = 0$. This reduction gives

$$\begin{array}{llll} a_2 = a_1 & b_2 = b_1 & p_{100} = q_{100} & q_{110} = -q_{10} \\ q_{111} = -q_{11} & p_{110} = -p_{10} & p_{111} = -p_{11} & \end{array} \tag{4.37}$$

and leaves q_1 and p_1 unrestricted. Thus we obtain the 6-wave system with the Hamiltonian

$$H^{(0)} = 2H_0(11) + 2H_0(10) + H_0(1) + 2 \left\{ \begin{matrix} 011 \\ 010, 001 \end{matrix} \right\} \tag{4.38}$$

which is related to the $\boldsymbol{A}_2$-subalgebra.

Example 19. $C_{19} = \mathbb{1}$. $U^*(\eta\lambda^*) + U(\lambda) = 0$. This reduction gives that all Cartan elements must be purely imaginary (real) for $\eta = 1$ ($\eta = -1$) and

$$p_\alpha^* = \eta p_\alpha \qquad q_\alpha^* = \eta q_\alpha. \tag{4.39}$$

Thus we get 12 'real' fields and 12-wave system with the Hamiltonian in general position with the upper restrictions. This reduction leads to the noncompact real form $sl(4, \mathbb{R})$ of the $\boldsymbol{A}_3$-algebra.

Example 20. $C_{20} = S_{e_1 - e_2}$. $C_{20}(U^*(\eta\lambda^*)) + U(\lambda) = 0$. Then

$$\begin{array}{llll} a_2 = -\eta a_1^* & a_{3,4}^* = -\eta a_{3,4} & b_2 = -\eta b_1^* & b_{3,4}^* = -\eta b_{3,4} \\ q_{110} = -\eta q_{10}^* & q_{111} = -\eta q_{11}^* & q_1 = \eta q_1^* & p_{100} = \eta q_{100}^* \\ p_{110} = -\eta p_{10}^* & p_{111} = -\eta p_{11}^* & p_1^* = \eta p_1. & \end{array} \tag{4.40}$$

This leads to the 7-wave (2 real and 5 complex) system with the Hamiltonian

$$H^{(0)} = -H_{0*}(100) + H_0(1) + 2\mathrm{Re}\, H_0(110) + 2\mathrm{Re}\, H_0(111) + \left\{ \begin{matrix} 111 \\ 011', 100 \end{matrix} \right\} + 2\mathrm{Re} \left\{ \begin{matrix} 111 \\ 110, 001 \end{matrix} \right\} + \left\{ \begin{matrix} 110 \\ 010', 100 \end{matrix} \right\}. \tag{4.41}$$

Example 21. $C_{21} = S_{e_1-e_2}S_{e_3-e_4}$. $C_{21}(U^*(\eta\lambda^*)) + U(\lambda) = 0$. Therefore

$$\begin{array}{llll} a_2 = -\eta a_1^* & a_4 = -\eta a_3^* & b_2 = -\eta b_1^* & b_4 = -\eta b_3^* \\ q_{110} = \eta q_{11}^* & q_{111} = \eta q_{10}^* & p_1 = \eta q_1^* & p_{100} = \eta q_{100}^* \\ p_{110} = \eta p_{11}^* & p_{111} = \eta p_{10}^* & & \end{array} \tag{4.42}$$

and we obtain the 6-wave (complex) system with the following Hamiltonian:

$$\begin{aligned} H^{(0)} &= -H_{0*}(100) - H_{0*}(1) + 2\mathrm{Re}\, H_0(111) + 2\mathrm{Re}\, H_0(11) \\ &\quad +2\mathrm{Re}\left(\left\{\begin{matrix} 111 \\ 011, 100 \end{matrix}\right\} + \left\{\begin{matrix} 111 \\ 110', 001 \end{matrix}\right\}\right). \end{aligned} \tag{4.43}$$

Example 22. $C_{22} = \mathbb{1}$. $C_{22}(U^T(-\lambda)) + U(\lambda) = 0$. This reduction does not restrict the Cartan elements. For the elements of the potential matrix we have the following restrictions:

$$p_\alpha = q_\alpha \tag{4.44}$$

and this leads to the 6-wave system

$$\begin{aligned} &\mathrm{i}(a_1 - a_2)q_{100,t} - \mathrm{i}(b_1 - b_2)q_{100,x} + \kappa_4 q_{11}q_{111} + \kappa_2 q_{10}q_{110} = 0 \\ &\mathrm{i}(a_2 - a_3)q_{10,t} - \mathrm{i}(b_2 - b_3)q_{10,x} + \kappa_2 q_{100}q_{110} - \kappa_3 q_1 q_{11} = 0 \\ &\mathrm{i}(a_3 - a_4)q_{1,t} - \mathrm{i}(b_3 - b_4)q_{1,x} + \kappa_1 q_{110}q_{111} - \kappa_3 q_{11}q_{10} = 0 \\ &\mathrm{i}(a_1 - a_3)q_{110,t} - \mathrm{i}(b_1 - b_3)q_{110,x} + \kappa_1 q_1 q_{111} - \kappa_2 q_{10}q_{100} = 0 \\ &\mathrm{i}(a_2 - a_4)q_{11,t} - \mathrm{i}(b_2 - b_4)q_{11,x} - \kappa_4 q_{100}q_{111} + \kappa_3 q_1 q_{10} = 0 \\ &\mathrm{i}(a_1 - a_4)q_{111,t} - \mathrm{i}(b_1 - b_4)q_{111,x} - \kappa_1 q_1 q_{110} - \kappa_4 q_{11}q_{100} = 0 \end{aligned} \tag{4.45}$$

where $\tilde{\kappa}_i$, $i = 1, \ldots, 4$ are given in appendix A. The Hamiltonian vanishes, see remark 3.

Example 23. $C_{23} = S_{e_3-e_4}$. $C_{23}(U^T(-\lambda)) + U(\lambda) = 0$. This gives

$$\begin{array}{llll} a_4 = a_3 & b_4 = b_3 & p_{100} = q_{100} & p_{110} = -q_{111} \\ p_{11} = -q_{10} & p_{10} = -q_{11} & p_{111} = -q_{110} & \end{array} \tag{4.46}$$

while the fields q_1 and p_1 are both unrestricted and redundant. This gives the 5-wave (complex) system

$$\begin{aligned} &\mathrm{i}(a_1 - a_2)q_{100,t} - \mathrm{i}(b_1 - b_2)q_{100,x} - \kappa_3(q_{10}q_{111} + q_{11}q_{110}) = 0 \\ &\mathrm{i}(a_2 - a_3)q_{10,t} - \mathrm{i}(b_2 - b_3)q_{10,x} + \kappa_3 q_{110}q_{100} = 0 \\ &\mathrm{i}(a_1 - a_3)q_{110,t} - \mathrm{i}(b_1 - b_3)q_{110,x} + \kappa_3 q_{10}q_{100} = 0 \\ &\mathrm{i}(a_2 - a_3)q_{11,t} - \mathrm{i}(b_2 - b_3)q_{11,x} + \kappa_3 q_{111}q_{100} = 0 \\ &\mathrm{i}(a_1 - a_3)q_{111,t} - \mathrm{i}(b_1 - b_3)q_{111,x} - \kappa_3 q_{11}q_{100} = 0. \end{aligned} \tag{4.47}$$

The Hamiltonian vanishes, see remark 3.

Example 24. $C_{24} = S_{e_2-e_3}S_{e_1-e_4}$. $C_{24}(U^T(-\lambda)) + U(\lambda) = 0$. Therefore

$$\begin{array}{llll} a_4 = -a_1 & a_3 = -a_2 & b_4 = -b_1 & b_3 = -b_2 \\ q_1 = -q_{100} & q_{11} = q_{110} & p_1 = -p_{100} & p_{11} = p_{110} \end{array} \tag{4.48}$$

and we obtain the 8-wave system with the Hamiltonian

$$H^{(0)} = H_0(1) + H_0(10) + H_0(11) + H_0(111) + 2\left(\left\{\begin{matrix} 111 \\ 011, 100' \end{matrix}\right\} + \left\{\begin{matrix} 011 \\ 010, 001 \end{matrix}\right\}\right).$$

Here p_{10}, q_{10} and p_{111}, q_{111} are unrestricted fields.

Example 25. $C_{25} = \mathbb{1}$. $C_{25}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$. This reduction gives that the Cartan elements must be real (purely imaginary) for $\eta = 1$ ($\eta = -1$) and for the potential matrix

$$p_\alpha = -\eta q_\alpha^* \qquad \alpha \in \Delta_+. \tag{4.49}$$

Thus we get the 6-wave system with the general Hamiltonian for this algebra and $\tilde{\kappa}_i$; $i = 1, \ldots, 4$ are real. The case $\eta = 1$ leads to the compact real form $su(4)$ for the $\boldsymbol{A}_3$-algebra.

Example 26. $C_{26} = \Sigma = \operatorname{diag}(s_1, s_2, s_3, s_4)$. $\Sigma U^\dagger(\eta\lambda^*)\Sigma^{-1} - U(\lambda) = 0$ and $s_1 s_2 s_3 s_4 = 1$. This reduction gives that the Cartan elements must be real (purely imaginary) for $\eta = 1$ ($\eta = -1$) and for the potential matrix

$$\begin{aligned} p_{100} &= -\eta \frac{s_2}{s_1} q_{100}^* \qquad & p_{10} &= -\eta \frac{s_3}{s_2} q_{10}^* \qquad & p_1 &= -\eta \frac{s_4}{s_3} q_1^* \\ p_{110} &= -\eta \frac{s_3}{s_1} q_{110}^* \qquad & p_{11} &= -\eta \frac{s_4}{s_2} q_{11}^* \qquad & p_{111} &= -\eta \frac{s_4}{s_1} q_{111}^*. \end{aligned} \tag{4.50}$$

Thus we get the 6-wave system with the Hamiltonian

$$\begin{aligned} H^{(0)} = {} & \frac{s_2}{s_1} H_{0*}(100) + \frac{s_3}{s_2} H_{0*}(10) + \frac{s_4}{s_3} H_{0*}(1) + \frac{s_3}{s_1} H_{0*}(110) + \frac{s_4}{s_2} H_{0*}(11) \\ & + \frac{s_4}{s_1} H_{0*}(111) + \frac{s_4}{s_2} \tilde{\kappa}_1 H_*(11, 1, 10) + \frac{s_4}{s_1} (\tilde{\kappa}_2 H_*(111, 110, 1) \\ & + \tilde{\kappa}_3 H_*(111, 11, 100)) + \frac{s_3}{s_1} \tilde{\kappa}_4 H_*(110, 10, 100) \end{aligned} \tag{4.51}$$

where $H_*(\alpha, \beta, \gamma)$ is given by (4.7) and $\tilde{\kappa}_i$; $i = 1, \ldots, 4$ are real. The case $\eta = 1$, $s_1 = s_2 = s_3 = s_4$ leads to the compact real form $su(4)$ for the $\boldsymbol{A}_3$-algebra, see the result of example 25. For $\eta = 1$ the choice $s_1 = -s_2 = -s_3 = -s_4$ gives us the noncompact real form $su(3, 1)$ and the choice $s_1 = s_2 = -s_3 = -s_4$ leads to another noncompact real form $su(2, 2)$ for the $\boldsymbol{A}_3$-algebra.

Example 27. $C_{27} = S_{e_3 - e_4}$. $C_{27}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$. Therefore

$$\begin{aligned} & a_{1,2}^* = \eta a_{1,2} \qquad & & a_4 = \eta a_3^* \qquad & & b_{1,2}^* = \eta b_{1,2} \qquad & & b_4 = \eta b_3^* \\ & p_{100} = -\eta q_{100}^* \qquad & & p_{111} = \eta q_{110}^* \qquad & & p_{110} = \eta q_{111}^* \qquad & & p_{10} = \eta q_{11}^* \\ & p_{11} = \eta q_{10}^* \qquad & & p_1^* = -\eta p_1 \qquad & & q_1^* = -\eta q_1 \end{aligned} \tag{4.52}$$

and we obtain the 7-wave (2 real and 5 complex) system with the Hamiltonian

$$\begin{aligned} H^{(0)} = {} & H_0(1) + H_{0*}(100) + 2\operatorname{Re} H_{0*}(11) + 2\operatorname{Re} H_{0*}(111) \\ & + 2\operatorname{Re}\left(\left\{ \begin{matrix} 111 \\ 011, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 011 \\ 010, 001 \end{matrix} \right\} \right). \end{aligned}$$

Example 28. $C_{28} = S_{e_2 - e_3} S_{e_1 - e_4}$. $C_{28}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$. Therefore

$$\begin{aligned} & a_4 = \eta a_1^* \qquad & & a_3 = \eta a_2^* \qquad & & b_4 = \eta b_1^* \qquad & & b_3 = \eta b_2^* \\ & q_1 = \eta q_{100}^* \qquad & & q_{10}^* = \eta q_{10} \qquad & & q_{11} = -\eta q_{110}^* \qquad & & q_{111}^* = \eta q_{111} \\ & p_1 = \eta p_{100}^* \qquad & & p_{10}^* = \eta p_{10} \qquad & & p_{11} = -\eta p_{110}^* \qquad & & p_{111}^* = \eta p_{111} \end{aligned} \tag{4.53}$$

and we obtain the 8-wave (4 real and 4 complex) system with the following Hamiltonian:

$$\begin{aligned} H^{(0)} = {} & H_0(10) + H_0(111) + 2\operatorname{Re} H_{0*}(100) + 2\operatorname{Re} H_{0*}(110) \\ & + 2\operatorname{Re}\left(\left\{ \begin{matrix} 111 \\ 110, 001' \end{matrix} \right\} + \left\{ \begin{matrix} 110 \\ 010, 100 \end{matrix} \right\} \right). \end{aligned} \tag{4.54}$$

This system is related to the noncompact real form $su^*(4)$ of $\boldsymbol{A}_3$.

4.5. $\mathfrak{g} \simeq \boldsymbol{B}_3 = so(7)$

In this case there are nine positive roots $\Delta_+ = \{100, 010, 001, 110, 011, 111, 012, 112, 122\}$ where again $ijk = i\alpha_1 + j\alpha_2 + k\alpha_3$ and $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3$. The interaction Hamiltonians below are given by (1.9) where the set of triples of indices $\mathcal{M}$ is {[122, 112, 10], [122, 111, 11], [122, 12, 110], [112, 111, 1], [112, 12, 100], [111, 110, 1], [111, 11, 100], [12, 11, 1], [11, 1, 10], [110, 10, 100]}.

Example 29. $C_{29} = w_0$. $C_{29}(U(-\lambda)) - U(\lambda) = 0$. This reduction does not restrict the Cartan elements. Therefore

$$p_\alpha = q_\alpha \tag{4.55}$$

thus we get 9-wave system. The Hamiltonian vanishes, see remark 3.

Example 30. $C_{30} = S_{e_1-e_2}$. $C_{30}(U(\lambda)) - U(\lambda) = 0$. Then

$$\begin{array}{llll} p_{100} = q_{100} & q_{110} = q_{10} & q_{111} = q_{11} & q_{112} = q_{12} \\ q_{122} = 0 & p_{110} = p_{10} & p_{111} = p_{11} & \\ p_{112} = p_{12} & p_{122} = 0 & a_2 = a_1 & b_2 = b_1. \end{array} \tag{4.56}$$

The interaction reduces to the 8-wave system with the Hamiltonian

$$H^{(0)} = 2H_0(10) + H_0(1) + 2H_0(11) + 2H_0(12) + 2\left(\left\{\begin{array}{c} 012 \\ 011, 001 \end{array}\right\} + \left\{\begin{array}{c} 011 \\ 010, 001 \end{array}\right\}\right). \tag{4.57}$$

In addition q_{100} becomes redundant, see remark 2, and p_1, q_1 are unrestricted ones. This system is related to the $\boldsymbol{B}_2$-subalgebra.

Example 31. $C_{31} = S_{e_3}$. $C_{31}(U(\lambda)) - U(\lambda) = 0$. Here we have

$$\begin{array}{llll} q_{112} = q_{110} & q_{12} = q_{10} & q_{111} = q_{11} = 0 & p_1 = -q_1 \\ p_{112} = p_{110} & p_{12} = p_{10} & p_{111} = p_{11} = 0 & a_3 = b_3 = 0. \end{array} \tag{4.58}$$

The Hamiltonian reduces to

$$H^{(0)} = H_0(100) + 2H_0(12) + 2H_0(112) + H_0(122) + 2\left(\left\{\begin{array}{c} 122 \\ 012, 110' \end{array}\right\} + \left\{\begin{array}{c} 112 \\ 012, 100 \end{array}\right\}\right). \tag{4.59}$$

Here q_1, p_1 are redundant fields and p_{100}, q_{100}, p_{122}, q_{122} are unrestricted. This system is related to the $\boldsymbol{D}_3$-subalgebra.

Example 32. $C_{32} = S_{e_1-e_2}S_{e_3}$. $C_{32}(U(-\lambda)) - U(\lambda) = 0$. Then

$$\begin{array}{lllll} p_{100} = -q_{100} & q_{12} = -q_{110} & q_{111} = q_{11} & q_{112} = -q_{10} & p_1 = q_1 \\ p_{12} = -p_{110} & p_{111} = p_{11} & p_{112} = -p_{10} & a_2 = -a_1 & b_2 = -b_1. \end{array} \tag{4.60}$$

However, this choice means that $C_{32}(J) = -J$ and therefore remark 3 applies. This automorphism reduces (1.1) to the following 8-wave equations:

$$\begin{aligned} &\mathrm{i}a_1 q_{100,t} - \mathrm{i}b_1 q_{100,x} + \kappa(q_{10}p_{110} - q_{110}p_{10}) = 0, \\ &\mathrm{i}(a_1 + a_3)q_{10,t} - \mathrm{i}(b_1 + b_3)q_{10,x} + 2\kappa(q_1 q_{11} - q_{100}q_{110}) = 0 \\ &\mathrm{i}a_3 q_{1,t} - \mathrm{i}b_3 q_{1,x} + \kappa(q_{11}p_{110} - q_{11}p_{10} + p_{11}q_{110} - p_{11}q_{10}) = 0 \\ &\mathrm{i}(a_1 - a_3)q_{110,t} - \mathrm{i}(b_1 - b_3)q_{110,x} + 2\kappa(q_1 q_{11} + q_{100}q_{10}) = 0 \\ &\mathrm{i}a_1 q_{11,t} - \mathrm{i}b_1 q_{11,x} - \kappa(q_1 q_{110} + q_1 q_{10}) = 0 \\ &\mathrm{i}(a_1 + a_3)p_{10,t} - \mathrm{i}(b_1 + b_3)p_{10,x} + 2\kappa(p_{11}q_1 + q_{100}p_{110}) = 0 \\ &\mathrm{i}(a_1 - a_3)p_{110,t} - \mathrm{i}(b_1 - b_3)p_{110,x} + 2\kappa(p_{11}q_1 - q_{100}p_{10}) = 0 \\ &\mathrm{i}a_1 p_{11,t} - \mathrm{i}b_1 p_{11,x} - \kappa(q_1 p_{110} + q_1 p_{10}) = 0 \end{aligned} \tag{4.61}$$

where $\kappa = a_1 b_3 - a_3 b_1$ and q_{122}, p_{122} are redundant, see remark 2.

Example 33. $C_{33} = S_{e_1}S_{e_2}$. $C_{33}(U(-\lambda)) - U(\lambda) = 0$. The reduction conditions give $C_{33}(J) = -J$ and

$$\begin{array}{llll} p_{100} = q_{100} & p_{112} = q_{110} & p_{111} = -q_{111} & p_{110} = q_{112} \\ p_1 = 0 & p_{122} = q_{122} & p_{12} = q_{10} & p_{11} = -q_{11} \\ p_{10} = q_{12} & q_1 = 0 & a_3 = 0 & b_3 = 0. \end{array} \qquad (4.62)$$

Again remark 3 applies and we obtain the next 8-wave system:

$$\begin{aligned} &\mathrm{i}(a_1 - a_2)q_{100,t} - \mathrm{i}(b_1 - b_2)q_{100,x} + \kappa(q_{10}q_{112} + q_{12}q_{110} - 2q_{11}q_{111}) = 0 \\ &\mathrm{i}a_2q_{10,t} - \mathrm{i}b_2q_{10,x} - \kappa(q_{100} + q_{122})q_{110} = 0 \\ &\mathrm{i}a_1q_{110,t} - \mathrm{i}b_1q_{110,x} - \kappa(q_{100} - q_{122})q_{10} = 0 \\ &\mathrm{i}a_2q_{11,t} - \mathrm{i}b_2q_{11,x} - \kappa(q_{100} + q_{122})q_{111} = 0 \\ &\mathrm{i}a_1q_{111,t} - \mathrm{i}b_1q_{111,x} - \kappa(q_{100} - q_{122})q_{11} = 0 \\ &\mathrm{i}a_2q_{12,t} - \mathrm{i}b_2q_{12,x} - \kappa(q_{100} + q_{122})q_{112} = 0 \\ &\mathrm{i}a_1q_{112,t} - \mathrm{i}b_1q_{112,x} - \kappa(q_{100} - q_{122})q_{12} = 0 \\ &\mathrm{i}(a_1 + a_2)q_{122,t} - \mathrm{i}(b_1 + b_2)q_{122,x} + \kappa(q_{10}q_{112} + q_{12}q_{110} - 2q_{11}q_{111}) = 0 \end{aligned} \qquad (4.63)$$

where $\kappa = a_1b_2 - a_2b_1$.

Example 34. $C_{34} = 1\!\!1$. $U^\dagger(\eta\lambda^*) - U(\lambda) = 0$, $\eta = \pm 1$. This reduction makes all Cartan elements real for $\eta = 1$ and purely imaginary for $\eta = -1$ and

$$p_\alpha = -\eta q_\alpha^*. \qquad (4.64)$$

Thus we get the 9-wave system with Hamiltonian (1.9) with the upper restrictions.

Example 35. $\Sigma = \mathrm{diag}\,(s_1, s_2, s_3, 1, 1/s_3, 1/s_2, 1/s_1)$, $\Sigma^{-1}U^\dagger(\eta\lambda^*)\Sigma - U(\lambda) = 0$, $\eta = \pm 1$. After this reduction all Cartan elements become real for $\eta = 1$ and purely imaginary for $\eta = -1$ and

$$\begin{array}{lll} p_{100} = -\eta\frac{s_1}{s_2}q_{100}^* & p_{10} = -\eta\frac{s_2}{s_3}q_{10}^* & p_1 = -\eta s_3q_1^* \\ p_{110} = -\eta\frac{s_1}{s_3}q_{110}^* & p_{11} = -\eta s_2q_{11}^* & p_{111} = -\eta s_1q_{111}^* \\ p_{12} = -\eta s_2s_3q_{12}^* & p_{112} = -\eta s_1s_3q_{112}^* & p_{122} = -\eta s_1s_2q_{122}^*. \end{array} \qquad (4.65)$$

Thus we get the 9-wave system with real Hamiltonian

$$\begin{aligned} H^{(0)} = {}& \frac{s_1}{s_2}H_{0*}(100) + \frac{s_2}{s_3}H_{0*}(10) + s_3H_{0*}(1) + \frac{s_1}{s_3}H_{0*}(110) + s_2H_{0*}(11) \\ &+s_1H_{0*}(111) + s_2s_3H_{0*}(12) + s_1s_3H_{0*}(112) + s_1s_2H_{0*}(122) \\ &+(\kappa_1 - \kappa_2 - \kappa_3)s_1s_2H_*(122, 112, 10) - \kappa_3s_1s_2H_*(122, 111, 11) \\ &+(\kappa_1 - \kappa_2 + \kappa_3)s_1s_2H_*(122, 12, 110) + \kappa_2s_1s_3H_*(112, 111, 1) \\ &+(-\kappa_1 - \kappa_2 + \kappa_3)s_1s_3H_*(112, 12, 100) + \kappa_2s_1H_*(111, 110, 1) \\ &+\kappa_3s_1H_*(111, 11, 100) - \kappa_1s_2s_3H_*(12, 11, 1) + \kappa_1s_2H_*(11, 1, 10) \\ &+(\kappa_1 + \kappa_2 + \kappa_3)\frac{s_1}{s_3}H_*(110, 10, 100) \end{aligned} \qquad (4.66)$$

where the terms $H_*(\alpha, \beta, \gamma)$ are given in (4.7). In the particular case $s_1 = s_2 = s_3 = 1$ the result coincides with example 34.

Example 36. $C_{36} = w_0$. $C_{36}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. This reduction requires that all fields p_α, q_α are real while all Cartan elements must be purely imaginary for $\eta = 1$ and vice versa for $\eta = -1$. Thus we get the 18-wave system with the Hamiltonian (1.9). The case with $\eta = 1$ leads to the real form $so(7, \mathbb{R}) \simeq so(7, 0)$ for the $\boldsymbol{B}_3$-algebra.

Example 37. $C_{37} = S_{e_1-e_2}$. $C_{37}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. Therefore

$$\begin{array}{llll} q^*_{100} = -\eta q_{100} & p_{10} = -\eta q^*_{110} & p_{11} = -\eta q^*_{111} & p_{12} = -\eta q^*_{112} \\ p^*_{100} = -\eta p_{100} & p_{110} = -\eta q^*_{10} & p_{111} = -\eta q^*_{11} & p_{112} = -\eta q^*_{12} \\ p_{122} = \eta q^*_{122} & p_1 = -\eta q^*_1 & a_2 = \eta a^*_1 & \\ b_2 = \eta b^*_1 & a^*_3 = \eta a_3 & b^*_3 = \eta b_3 & \end{array} \tag{4.67}$$

which gives the 10–wave (2 real and 8 complex) system with the following Hamiltonian:

$$\begin{aligned} H^{(0)} &= H_0(100) + H_{0*}(1) + H_{0*}(122) + 2\mathrm{Re}\, H_{0*}(110) \\ &\quad + \left\{ \begin{matrix} 122 \\ 111, 011 \end{matrix} \right\} + \left\{ \begin{matrix} 112 \\ 012, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 011, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 110 \\ 010, 100 \end{matrix} \right\} \\ &\quad + 2\mathrm{Re} \left(\left\{ \begin{matrix} 122 \\ 012, 110 \end{matrix} \right\} + \left\{ \begin{matrix} 112 \\ 111, 001 \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 110, 001 \end{matrix} \right\} \right). \end{aligned} \tag{4.68}$$

Example 38. $C_{38} = S_{e_3}$. $C_{38}(U^\dagger(\eta\lambda^*)) - U(\lambda)$, $\eta = \pm 1$. Then

$$\begin{array}{llll} p_{100} = -\eta q^*_{100} & p_{112} = -\eta q^*_{110} & p_{111} = \eta q^*_{111} & p_{110} = -\eta q^*_{112} \\ p_{122} = -\eta q^*_{122} & p_{12} = -\eta q^*_{10} & p_{11} = \eta q^*_{11} & p_{10} = -\eta q^*_{12} \\ q^*_1 = \eta q_1 & p^*_1 = \eta p_1 & a^*_3 = -\eta a_3 & b^*_3 = -\eta b_3 \\ a^*_1 = \eta a_1 & a^*_2 = \eta a_2 & b^*_1 = \eta b_1 & b^*_2 = \eta b_2 \end{array} \tag{4.69}$$

so we obtain the 10-wave (2 real and 8 complex) system with the Hamiltonian

$$\begin{aligned} H^{(0)} &= H_{0*}(100) + H_{0*}(11) + H_{0*}(111) + H_{0*}(122) + H_0(1) + 2\mathrm{Re}\, H_{0*}(112) \\ &\quad + \left\{ \begin{matrix} 122 \\ 111, 011 \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 011, 100 \end{matrix} \right\} + 2\mathrm{Re} \left(\left\{ \begin{matrix} 122 \\ 012, 110 \end{matrix} \right\} + \left\{ \begin{matrix} 110 \\ 010, 100 \end{matrix} \right\} \right. \\ &\quad \left. + \left\{ \begin{matrix} 111 \\ 110, 001 \end{matrix} \right\} + \left\{ \begin{matrix} 012 \\ 011, 001 \end{matrix} \right\} \right). \end{aligned} \tag{4.70}$$

Example 39. $C_{39} = S_{e_1-e_2} S_{e_3}$. $C_{39}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. Then

$$\begin{array}{llll} q_{100} = -\eta q^*_{100} & p_{12} = -\eta q^*_{110} & p_{11} = \eta q^*_{111} & p_{10} = -\eta q^*_{112} \\ p_{122} = \eta q^*_{122} & p_{112} = -\eta q^*_{10} & p_{111} = \eta q^*_{11} & p_{110} = -\eta q^*_{12} \\ q^*_1 = \eta q_1 & p^*_1 = \eta p_1 & p^*_{100} = -\eta p_{100} & \\ a^*_3 = -\eta a_3 & b^*_3 = -\eta b_3 & a_2 = \eta a^*_1 & b_2 = \eta b^*_1 \end{array} \tag{4.71}$$

and we obtain the 11-wave (4 real and 7 complex) system with Hamiltonian

$$\begin{aligned} H^{(0)} &= H_0(100) + H_0(1) + H_{0*}(122) + 2\mathrm{Re}\, H_{0*}(112) + 2\mathrm{Re}\, H_{0*}(12) + 2\mathrm{Re}\, H_{0*}(111) \\ &\quad + \left\{ \begin{matrix} 122 \\ 012, 110 \end{matrix} \right\} + \left\{ \begin{matrix} 122 \\ 111, 011 \end{matrix} \right\} + \left\{ \begin{matrix} 122 \\ 112, 010 \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 011, 100 \end{matrix} \right\} \\ &\quad + 2\mathrm{Re} \left(\left\{ \begin{matrix} 110 \\ 010, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 112 \\ 111, 001 \end{matrix} \right\} + \left\{ \begin{matrix} 012 \\ 011, 001 \end{matrix} \right\} \right). \end{aligned} \tag{4.72}$$

Example 40. $C_{40} = S_{e_1} S_{e_2}$. $C_{40}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$. As a result

$$\begin{array}{lllll} a^*_3 = \eta a_3 & b^*_3 = \eta b_3 & a^*_{1,2} = -\eta a_{1,2} & b^*_{1,2} = -\eta b_{1,2} & \\ q_{112} = \eta q^*_{110} & q_{12} = \eta q^*_{10} & p_{112} = \eta p^*_{110} & p_{12} = \eta p^*_{10} & \\ q^*_{100} = \eta q_{100} & p^*_{100} = \eta p_{100} & q^*_{111} = -\eta q_{111} & p^*_{111} = -\eta p_{111}, & \\ q^*_{122} = \eta q_{122} & p^*_{122} = \eta p_{122} & p_1 = -\eta q^*_1 & q^*_{11} = -\eta q_{11} & p^*_{11} = -\eta p_{11} \end{array} \tag{4.73}$$

which leads to the 13-wave (8 real and 5 complex) system with Hamiltonian

$$H^{(0)} = H_0(100) + H_0(122) + H_{0*}(1) + H_{0*}(11) + H_{0*}(111) + 2\mathrm{Re}\, H_{0*}(12) + 2\mathrm{Re}\, H_{0*}(112) + \left\{ \begin{matrix} 122 \\ 111, 011 \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 011, 100 \end{matrix} \right\} + 2\mathrm{Re} \left(\left\{ \begin{matrix} 122 \\ 012, 110' \end{matrix} \right\} + \left\{ \begin{matrix} 112 \\ 012, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 112 \\ 111, 001 \end{matrix} \right\} + \left\{ \begin{matrix} 012 \\ 011, 001 \end{matrix} \right\} \right). \tag{4.74}$$

4.6. $\mathfrak{g} \simeq \boldsymbol{C}_3 = sp(6)$

In this case there are nine positive roots $\Delta_+ = \{100, 010, 001, 110, 011, 111, 021, 121, 221\}$ where again $ijk = i\alpha_1 + j\alpha_2 + k\alpha_3$ and $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = 2e_3$, the set of triples of indices is $\mathcal{M} \equiv \{[110, 10, 100], [111, 11, 100], [121, 21, 100], [121, 11, 110], [21, 11, 10], [111, 110, 1], [11, 1, 10], [121, 111, 10], [221, 121, 100], [221, 111, 110]\}$.

Example 41. $C_{41} = w_0$. $C_{41}(U(-\lambda)) - U(\lambda) = 0$. This reduction does not restrict the Cartan elements. Therefore

$$p_\alpha = q_\alpha \qquad \alpha \in \Delta \tag{4.75}$$

and leads to the 9-wave system with vanishing Hamiltonian, see remark 3. This system is similar to the general one without reductions but with the fields p_α replaced by q_α.

Example 42. $C_{42} = S_{e_1 - e_2}$. $C_{42}(U(\lambda)) - U(\lambda) = 0$. Therefore

$$\begin{aligned} &q_{110} = q_{10} \qquad q_{111} = q_{11} \qquad q_{221} = -q_{21} \qquad p_{100} = q_{100} \\ &p_{110} = p_{10} \qquad p_{111} = p_{11} \qquad p_{221} = -p_{21} \qquad a_2 = a_1 \qquad b_2 = b_1 \end{aligned} \tag{4.76}$$

and we obtain the 10-wave system which is described by the Hamiltonian

$$H^{(0)} = 2H_0(110) + H_0(1) + 2H_0(111) + 2H_0(221) + H_0(121) + 2 \left(\left\{ \begin{matrix} 221 \\ 111, 110 \end{matrix} \right\} + \left\{ \begin{matrix} 121 \\ 011', 110 \end{matrix} \right\} + \left\{ \begin{matrix} 011' \\ 010', 001 \end{matrix} \right\} \right). \tag{4.77}$$

Note that the functions q_{121}, p_{121} and q_1, p_1 remain unrestricted and q_{100} is redundant, see remark 2.

Example 43. $C_{43} = S_{2e_3}$. $C_{43}(U(\lambda)) - U(\lambda) = 0$. Then

$$\begin{aligned} &q_{111} = -q_{110} \qquad q_{11} = -q_{10} \qquad p_{111} = -p_{110} \qquad p_{11} = -p_{10} \\ &p_{221} = p_{121} = p_{21} = 0 \qquad q_{221} = q_{121} = q_{21} = 0 \qquad p_1 = q_1 \\ &a_3 = 0 \qquad b_3 = 0 \end{aligned} \tag{4.78}$$

giving the 6-wave (complex) system with the Hamiltonian

$$H^{(0)} = H_0(100) + 2H_0(10) + 2H_0(110) + 2 \left\{ \begin{matrix} 110 \\ 010, 100 \end{matrix} \right\} \tag{4.79}$$

related to $\boldsymbol{A}_2$-subalgebra. Here $\kappa = a_1 b_2 - a_2 b_1$, q_{100} and p_{100} are unrestricted fields and q_1 is redundant, see remark 2.

Example 44. $C_{44} = S_{e_1 - e_2} S_{2e_3}$. $C_{44}(U(-\lambda)) - U(\lambda) = 0$. This gives

$$\begin{aligned} &a_2 = -a_1 \qquad b_2 = -b_1 \qquad p_{100} = -q_{100} \qquad q_{110} = q_{11} \qquad q_{111} = q_{10} \\ &q_{221} = -q_{21} \qquad p_1 = -q_1 \qquad p_{110} = p_{11} \qquad p_{111} = p_{10} \qquad p_{221} = -p_{21} \end{aligned} \tag{4.80}$$

and the next 8-wave system, see remark 3:

$$\begin{aligned}
&\mathrm{i}a_1q_{100,t} - \mathrm{i}b_1q_{100,x} + \kappa(p_{10}q_{11} - p_{11}q_{10}) = 0\\
&\mathrm{i}(a_1+a_3)q_{10,t} - \mathrm{i}(b_1+b_3)q_{10,x} - 2\kappa(q_{21}p_{11} + q_{100}q_{11} + q_1q_{11}) = 0\\
&\mathrm{i}a_3q_{1,t} - \mathrm{i}b_3q_{1,x} - \kappa(p_{10}q_{11} - p_{11}q_{10}) = 0\\
&\mathrm{i}(a_1-a_3)q_{11,t} - \mathrm{i}(b_1-b_3)q_{11,x} - 2\kappa(q_{21}p_{10} - q_{100}q_{10} + q_1q_{10}) = 0\\
&\mathrm{i}a_1q_{21,t} - \mathrm{i}b_1q_{21,x} - 2\kappa q_{10}q_{11} = 0\\
&\mathrm{i}(a_1+a_3)p_{10,t} - \mathrm{i}(b_1+b_3)p_{10,x} + 2\kappa(q_1p_{11} + q_{100}p_{11} - p_{21}q_{11}) = 0\\
&\mathrm{i}(a_1-a_3)p_{11,t} - \mathrm{i}(b_1-b_3)p_{11,x} + 2\kappa(q_1p_{10} - q_{100}p_{10} - p_{21}q_{10}) = 0\\
&\mathrm{i}a_1p_{21,t} - \mathrm{i}b_1p_{21,x} + 2\kappa p_{10}p_{11} = 0
\end{aligned} \tag{4.81}$$

where $\kappa = a_1b_3 - a_3b_1$ and q_{121} and p_{121} are redundant fields, see remark 2.

Example 45. $C_{45} = S_{2e_1}S_{2e_3}$. $C_{45}(U(-\lambda)) - U(\lambda) = 0$. Then

$$\begin{array}{lllll}
a_2 = 0 & b_2 = 0 & p_{121} = -q_{100} & p_{100} = -q_{121} & p_{111} = \mathrm{i}q_{111}\\
p_{110} = \mathrm{i}q_{110} & p_{221} = -q_{221} & p_1 = q_1 & q_{11} = \mathrm{i}q_{10} & p_{11} = -\mathrm{i}p_{10}.
\end{array} \tag{4.82}$$

So we get the next 8-wave system, see remark 3:

$$\begin{aligned}
&\mathrm{i}a_1q_{100,t} - \mathrm{i}b_1q_{100,x} - \kappa(q_{111}p_{10} + \mathrm{i}p_{10}q_{110}) = 0\\
&\mathrm{i}a_3q_{10,t} - \mathrm{i}b_3q_{10,x} + \kappa(q_{121}q_{111} + \mathrm{i}q_{121}q_{110}) = 0\\
&\mathrm{i}a_3q_{1,t} - \mathrm{i}b_3q_{1,x} + 2\mathrm{i}\kappa q_{110}q_{111} = 0\\
&\mathrm{i}(a_1-a_3)q_{110,t} - \mathrm{i}(b_1-b_3)q_{110,x} + \kappa(2\mathrm{i}q_{221}q_{111} + \mathrm{i}q_{121}p_{10} + 2q_1q_{111} + q_{100}q_{10}) = 0\\
&\mathrm{i}(a_1+a_3)q_{111,t} - \mathrm{i}(b_1+b_3)q_{111,x} + \kappa(2\mathrm{i}q_{221}q_{110} - 2q_1q_{110} - \mathrm{i}q_{100}q_{10} - q_{121}p_{10}) = 0\\
&\mathrm{i}a_1q_{121,t} - \mathrm{i}b_1q_{121,x} + \kappa q_{111}q_{10} = 0\\
&\mathrm{i}a_1q_{221,t} - \mathrm{i}b_1q_{221,x} - 2\kappa(q_{111}q_{110} + \mathrm{i}q_{110}q_{10}) = 0\\
&\mathrm{i}a_3p_{10,t} - \mathrm{i}b_3p_{10,x} + \kappa(\mathrm{i}q_{100}q_{110} - q_{100}q_{111}) = 0
\end{aligned} \tag{4.83}$$

where $\kappa = a_1b_3 - a_3b_1$ and q_{21} and p_{21} are redundant fields, see remark 2.

Example 46. $C_{46} = 1\!\!1$. $U^{\dagger}(\eta\lambda^*) - U(\lambda) = 0$, $\eta = \pm 1$. This reduction means that all Cartan elements are real for $\eta = 1$ and purely imaginary for $\eta = -1$ and

$$p_{\alpha} = -\eta q_{\alpha}^*. \tag{4.84}$$

Thus we get the 9-wave system with Hamiltonian (1.9) with the restrictions given above.

Example 47. $\Sigma = \operatorname{diag}(s_1, s_2, s_3, 1/s_3, 1/s_2, 1/s_1)$, $\Sigma^{-1}U^{\dagger}(\eta\lambda^*)\Sigma - U(\lambda) = 0$. After this reduction all Cartan elements are real for $\eta = 1$ and purely imaginary for $\eta = -1$ and

$$\begin{array}{lll}
p_{100} = -\eta\frac{s_1}{s_2}q_{100}^* & p_{10} = -\eta\frac{s_2}{s_3}q_{10}^* & p_1 = -\eta s_3^2q_1^*\\
p_{110} = -\eta\frac{s_1}{s_3}q_{110}^* & p_{11} = -\eta s_2s_3q_{11}^* & p_{111} = -\eta s_1s_3q_{111}^*\\
p_{21} = -\eta s_2^2q_{12}^* & p_{121} = -\eta s_1s_2q_{121}^* & p_{221} = -\eta s_1^2q_{221}^*
\end{array} \tag{4.85}$$

which leads to a 9-wave system with the Hamiltonian

$$\begin{aligned}
H^{(0)} = {}& \frac{s_1}{s_2}H_{0*}(100) + \frac{s_2}{s_3}H_{0*}(10) + s_3^2H_{0*}(1) + \frac{s_1}{s_3}H_{0*}(110) + s_2s_3H_{0*}(11)\\
&+ s_1s_3H_{0*}(111) + s_2^2H_{0*}(21) + s_1s_2H_{0*}(121) + s_1^2H_{0*}(221)\\
&+ (\kappa_1+\kappa_2+\kappa_3)\frac{s_1}{s_3}H_*(110, 10, 100)\\
&+ (-\kappa_1-\kappa_2+\kappa_3)s_1s_3H_*(111, 11, 100)
\end{aligned}$$

$$+2\kappa_3 s_1 s_2 H_*(121, 21, 100) + (\kappa_1 - \kappa_2 + \kappa_3) s_1 s_2 H_*(121, 11, 110)$$
$$+2\kappa_1 s_2^2 H_*(21, 11, 10) + 2\kappa_2 s_1 s_3 H_*(111, 110, 1) + 2\kappa_1 s_2 s_3 H_*(11, 1, 10)$$
$$+(\kappa_1 - \kappa_2 - \kappa_3) s_1 s_2 H_*(121, 111, 10) + 2\kappa_3 s_1^2 H_*(221, 121, 100)$$
$$-2\kappa_2 s_1^2 H_*(221, 111, 110) \qquad (4.86)$$

where the terms $H_*(\alpha, \beta, \gamma)$ are given in (4.7). In the particular case $s_1 = s_2 = s_3 = 1$ we obtain the result of example 46.

Example 48. $C_{48} = w_0$. $C_{48}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. This reduction means that all fields q_α, p_α are real and the Cartan elements are purely imaginary for $\eta = 1$ and vice versa for $\eta = -1$. Thus we get the 18-wave system with the Hamiltonian (1.9). The case $\eta = 1$ leads to the noncompact real form $sp(6, \mathbb{R})$ for the $\boldsymbol{C}_3$-algebra.

Example 49. $C_{49} = S_{e_1-e_2}$. $C_{49}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. Therefore

$$\begin{array}{llll} q_{100}^* = -\eta q_{100} & p_{100}^* = -\eta p_{100} & p_1 = -\eta q_1^* & p_{121} = -\eta q_{121}^* \\ p_{10} = -\eta q_{110}^* & p_{110} = -\eta q_{10}^* & p_{11} = -\eta q_{111}^* & p_{111} = -\eta q_{11}^* \\ p_{21} = \eta q_{221}^* & p_{221} = \eta q_{21}^* & a_2 = \eta a_1^* & \\ b_2 = \eta b_1^* & a_3^* = \eta a_3 & b_3^* = \eta b_3. & \end{array} \qquad (4.87)$$

So this leads to the 10-wave (2 real and 8 complex) system with the Hamiltonian

$$H^{(0)} = H_0(100) + H_{0*}(121) + H_{0*}(1) + 2\mathrm{Re}\, H_{0*}(221) + 2\mathrm{Re}\, H_{0*}(111) + 2\mathrm{Re}\, H_{0*}(110)$$
$$+ \left\{ \begin{array}{c} 111 \\ 011, 100 \end{array} \right\} + \left\{ \begin{array}{c} 110 \\ 010, 100 \end{array} \right\} + 2\mathrm{Re} \left(\left\{ \begin{array}{c} 221 \\ 121, 100 \end{array} \right\} + \left\{ \begin{array}{c} 221 \\ 111, 110 \end{array} \right\} \right.$$
$$\left. + \left\{ \begin{array}{c} 121 \\ 011, 110 \end{array} \right\} + \left\{ \begin{array}{c} 011 \\ 010, 001 \end{array} \right\} \right). \qquad (4.88)$$

Example 50. $C_{50} = S_{2e_3}$. $C_{50}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. Then

$$\begin{array}{llll} q_1^* = -\eta q_1 & p_1^* = -\eta p_1 & p_{100} = -\eta q_{100}^* & p_{111} = \eta q_{110}^* \\ p_{110} = \eta q_{111}^* & p_{11} = \eta q_{10}^* & p_{10} = \eta q_{11}^* & p_{121} = \eta q_{121}^* \\ p_{221} = \eta q_{221}^* & p_{21} = \eta q_{21}^* & a_{1,2}^* = \eta a_{1,2} & b_{1,2}^* = \eta b_{1,2} \\ a_3^* = -\eta a_3 & b_3^* = -\eta b_3. & & \end{array} \qquad (4.89)$$

Thus we obtain the 10-wave (2 real and 8 complex) system with the Hamiltonian

$$H^{(0)} = H_{0*}(100) + H_0(1) + H_{0*}(21) + H_{0*}(121) + H_{0*}(221) + 2\mathrm{Re}\, H_{0*}(111) + 2\mathrm{Re}\, H_{0*}(111)$$
$$+ \left\{ \begin{array}{c} 221 \\ 121, 100 \end{array} \right\} + \left\{ \begin{array}{c} 221 \\ 111, 110 \end{array} \right\} + \left\{ \begin{array}{c} 121 \\ 021, 100 \end{array} \right\} + \left\{ \begin{array}{c} 111 \\ 110, 001 \end{array} \right\}$$
$$+ \left\{ \begin{array}{c} 021 \\ 011, 010 \end{array} \right\} + \left\{ \begin{array}{c} 011 \\ 010, 001 \end{array} \right\} + 2\mathrm{Re} \left(\left\{ \begin{array}{c} 121 \\ 011, 110 \end{array} \right\} + \left\{ \begin{array}{c} 111 \\ 011, 100 \end{array} \right\} \right). \qquad (4.90)$$

Example 51. $C_{51} = S_{e_1-e_2} S_{2e_3}$. $C_{51}(U^\dagger(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. Then

$$\begin{array}{llll} p_{100}^* = -\eta p_{100} & q_{100}^* = -\eta q_{100} & p_{11} = \eta q_{110}^* & \\ p_{111} = \eta q_{10}^* & p_{10} = \eta q_{111}^* & & \\ p_{110} = \eta q_{11}^* & p_{121} = \eta q_{121}^* & p_{21} = -\eta q_{221}^* & p_{221} = -\eta q_{21}^* \\ q_1^* = -\eta q_1 & p_1^* = -\eta p_1 & a_2 = \eta a_1^* & \\ b_2 = \eta b_1^* & a_3^* = -\eta a_3 & b_3^* = -\eta b_3 & \end{array} \qquad (4.91)$$

giving the 11-wave (4 real and 7 complex) system with Hamiltonian

$$H^{(0)} = H_0(100) + H_0(1) + H_{0*}(121) + 2\mathrm{Re}\, H_{0*}(11) + 2\mathrm{Re}\, H_{0*}(111) + 2\mathrm{Re}\, H_{0*}(221) + \left\{ \begin{matrix} 121 \\ 011, 110 \end{matrix} \right\} + \left\{ \begin{matrix} 121 \\ 111, 010 \end{matrix} \right\} + 2\mathrm{Re}\left(\left\{ \begin{matrix} 221 \\ 121, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 221 \\ 111, 110 \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 110, 001 \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 011, 100 \end{matrix} \right\} \right). \tag{4.92}$$

Example 52. $C_{52} = S_{e_1-e_2} S_{e_1+e_2}$. $C_{52}(U^{\dagger}(\eta\lambda^*)) - U(\lambda) = 0$, $\eta = \pm 1$. Therefore

$$\begin{aligned} &q_{100}^* = -\eta q_{100} \quad q_{111} = \eta \mathrm{i} q_{110}^* \quad q_{11} = -\eta \mathrm{i} q_{10}^* \quad q_{21}^* = \eta q_{21} \\ &q_{121}^* = -\eta q_{121} \quad q_{221}^* = \eta q_{221} \quad p_{100}^* = -\eta p_{100} \quad p_{111} = -\eta \mathrm{i} p_{110}^* \\ &p_{11} = \eta \mathrm{i} p_{10}^* \quad p_{21}^* = \eta p_{21} \quad p_{121}^* = -\eta p_{121} \quad p_{221}^* = \eta p_{221} \quad p_1 = \eta q_1^* \\ &a_{1,2}^* = -\eta a_{1,2} \quad b_{1,2}^* = -\eta b_{1,2} \quad a_3^* = \eta a_3 \quad b_3^* = \eta b_3. \end{aligned} \tag{4.93}$$

Thus we get the 13-wave (8 real and 5 complex) system with Hamiltonian

$$H^{(0)} = H_0(100) + H_0(121) + H_0(21) + H_{0*}(1) + 2\mathrm{Re}\, H_{0*}(11) + 2\mathrm{Re}\, H_{0*}(111) + H_0(221) + \left\{ \begin{matrix} 221 \\ 121, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 121 \\ 021, 100 \end{matrix} \right\} + \left\{ \begin{matrix} 021 \\ 011, 010' \end{matrix} \right\} + \left\{ \begin{matrix} 011 \\ 010', 001 \end{matrix} \right\} + \left\{ \begin{matrix} 221 \\ 111, 110' \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 110', 001 \end{matrix} \right\} + 2\mathrm{Re}\left(\left\{ \begin{matrix} 121 \\ 111, 010' \end{matrix} \right\} + \left\{ \begin{matrix} 111 \\ 011, 100 \end{matrix} \right\} \right). \tag{4.94}$$

5. Real forms of $\mathfrak{g}$ as $\mathbb{Z}_2$-reductions

As already mentioned, in several of the examples above the $\mathbb{Z}_2$-reductions act as Cartan involutions, i.e. $\mathrm{i}U(x,\lambda)$ belongs to a real form $\mathfrak{g}^{\mathbb{R}}$ of the corresponding complex simple Lie algebra $\mathfrak{g}$. As a result, the scattering matrix $T(\lambda)$ (see equation (6.2) below) belongs to the corresponding compact or noncompact Lie group.

It is well known that $X \in \mathfrak{g}^{\mathbb{R}}$ if $X \in \mathfrak{g}$ and (see, for example, [18])

$$\sigma(\theta(X)) \equiv \theta(\sigma(X)) = X \qquad \theta(X) = -X^{\dagger} \qquad X \in \mathfrak{g}$$

where σ is an involutive Cartan automorphism: $\sigma^2 = \mathbb{1}$. The related $\mathbb{Z}_2$-reduction acts in addition on the complex spectral parameter λ via complex conjugation: $\kappa(\lambda) = \lambda^*$. The compact real form $\tilde{\mathfrak{g}}^{\mathbb{R}}$ of $\mathfrak{g}$ is obtained with $\sigma = \mathbb{1}$. For the noncompact cases the Cartan involution splits the roots of $\mathfrak{g}$ into compact and noncompact ones as follows:

(1) If $\sigma(E_\alpha) = E_\alpha$, where E_α is the Weyl generator for the root α, we say that α is a compact root.

The noncompact roots are of two types depending on the orbit-size of σ:

(2) If $\sigma(E_\alpha) = \varepsilon E_\alpha$, $\varepsilon = \pm 1$ the orbit of σ consist of only one element;

(3) If $\sigma(E_\alpha) = \varepsilon E_{-\beta}$, $\alpha \neq \beta > 0$ and $\varepsilon = \pm 1$ then $\{\alpha, \beta\}$ is a two-element orbit of σ.

Let π be the system of simple roots of the algebra and π_0 be the set of the compact simple roots. The (inner) Cartan involution which extracts the noncompact real form $\mathfrak{g}^{\mathbb{R}}$ from the compact one is given by

$$\sigma = \exp\left(\sum_{\alpha_k \in \pi \setminus \pi_0} \frac{2\pi \mathrm{i}}{(\alpha_k, \alpha_k)} H_{\omega_k} \right). \tag{5.1}$$

Here $H_{\omega_k} = \sum_{i=1}^{r} (\omega_k, e_i) h_i$, where $\{h_i\}$ is the basis in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ dual to the orthogonal basis $\{e_i\}$ in the root space and $\{\omega_k\}$ are the fundamental weights of the algebra.

Table 3. List of reductions related to real forms of $\mathfrak{g}$. In all examples we assume $\eta = 1$.

$\mathfrak{g}$	Compact real form	Noncompact real form	Cartan involution
$\boldsymbol{A}_2$	$su(3)$, Ex. 4 and	$sl(3,\mathbb{R})$, Ex. 2	External
	Ex. 6, $s_1 = s_2 = s_3$.	$su(2,1)$, Ex. 6	$s_1 = -s_2 = -s_3$
$\boldsymbol{C}_2$	$sp(4)$, Ex. 8 and	$sp(4,\mathbb{R})$, Ex. 7	
	Ex. 9, $s_1 = s_2 = 1$.	$sp(2,2)$, Ex. 9	$s_1 = -1, s_2 = 1$
$\boldsymbol{G}_2$	$\mathfrak{g}_2$, Ex. 14 and	$\mathfrak{g}_2'$, Ex. 15	$s_1 = s_2 = -1$
	Ex. 15, $s_1 = s_2 = 1$.		
$\boldsymbol{A}_3$	$su(4)$, Ex. 25 and	$sl(4,\mathbb{R})$, Ex. 19	External
	Ex. 26, $s_1 = s_2 = s_3 = s_4$.	$su(1,3)$, Ex. 26	$s_1 = -s_2 = -s_3 = -s_4$
		$su(2,2)$, Ex. 26	$s_1 = s_2 = -s_3 = -s_4$
		$su^*(4)$, Ex. 28	$\sigma = S_{e_2-e_3} \circ S_{e_1-e_4}$
$\boldsymbol{B}_3$	$so(7) \simeq so(7,\mathbb{R})$,	$so(2,5)$, Ex. 35	$s_1 = -1, s_2 = s_3 = 1$
	Ex. 34, Ex. 36 and	$so(3,4)$, Ex. 35	$s_1 = s_2 = -1, s_3 = 1$
	Ex. 35, $s_1 = s_2 = s_3 = 1$.	$so(1,6)$, Ex. 35	$s_1 = s_2 = s_3 = -1$
$\boldsymbol{C}_3$	$sp(6)$, Ex. 46 and	$sp(6,\mathbb{R})$, Ex. 48	
	Ex. 47, $s_1 = s_2 = s_3 = 1$.	$sp(2,4)$, Ex.47	$s_1 = -s_2 = -s_3 = 1$

Note that in the examples above for the $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ algebras the corresponding normal real forms $sl(3,\mathbb{R})$, $sl(4,\mathbb{R})$ and $su^*(4)$ are extracted with external automorphisms. We leave more details about noncompact real forms generated by external involutive automorphisms to the second paper of this sequence.

The list of the Cartan involutions for the considered real forms of the simple Lie algebras and the relevant examples is given in table 3.

6. Scattering data and the $\mathbb{Z}_2$-reductions

In order to determine the scattering data of the Lax operator (1.2) we start from the Jost solutions

$$\lim_{x\to\infty} \psi(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = 1\!\!1 \qquad \lim_{x\to-\infty} \phi(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = 1\!\!1 \tag{6.1}$$

and the scattering matrix

$$T(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda). \tag{6.2}$$

Let us start with the simplest case when J has purely real and pair-wise distinct eigenvalues. Following the classical papers of Zakharov and Shabat [7, 20], the most efficient way to construct the minimal set of scattering data of (1.2) and to study its properties is to make use of the equivalent RHP.

We treat below only the simplest non-trivial case when J has real pair-wise distinct eigenvalues, i.e. when $(a,\alpha_j) > 0$ for $j = 1,\ldots,r$, see [8]. Then one is able to construct the fundamental analytical solutions (FAS) of (1.2) $\chi^{\pm}(x,\lambda)$ by using the Gauss decomposition of $T(\lambda)$:

$$T(\lambda) = T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda) \tag{6.3}$$

where by 'hat' above we denote the inverse matrix $\hat{S} \equiv S^{-1}$ and

$$S^{\pm}(\lambda) = \exp\left(\sum_{\alpha\in\Delta_+} s_{\pm}^{\pm\alpha}(\lambda)E_{\pm\alpha}\right) \qquad T^{\pm}(\lambda) = \exp\left(\sum_{\alpha\in\Delta_+} t_{\pm}^{\pm\alpha}(\lambda)E_{\pm\alpha}\right) \tag{6.4}$$

$$D^+(\lambda) = I\exp\left(\sum_{j=1}^{r}\frac{2d_j^+(\lambda)}{(\alpha_j,\alpha_j)}H_j\right) \qquad D^-(\lambda) = I\exp\left(\sum_{j=1}^{r}\frac{2d_j^-(\lambda)}{(\alpha_j,\alpha_j)}H_j^-\right) \tag{6.5}$$

$$H_j \equiv H_{\alpha_j} \qquad H_j^- = w_0(H_j).$$

Here I is an element from the universal centre of $\mathfrak{G}$ and the superscript + (or $-$) in $D^{\pm}(\lambda)$ shows that $D_j^{+}(\lambda)$ (or $D_j^{-}(\lambda)$) are analytic functions of λ for $\operatorname{Im}\lambda > 0$ (or $\operatorname{Im}\lambda < 0$ respectively). Then we can prove that [8]

$$\chi^{\pm}(x,\lambda) = \phi(x,\lambda)S^{\pm}(\lambda) = \psi(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda) \tag{6.6}$$

are FAS of (1.2) for $\operatorname{Im}\lambda \gtrless 0$. On the real axis $\chi^{+}(x,\lambda)$ and $\chi^{-}(x,\lambda)$ are linearly related by

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G_0(\lambda) \qquad G_0(\lambda) = S^{+}(\lambda)\hat{S}^{-}(\lambda) \tag{6.7}$$

and the sewing function $G_0(\lambda)$ may be considered as a minimal set of scattering data provided the Lax operator (1.2) has no discrete eigenvalues. The presence of discrete eigenvalues $\lambda_k^{\pm}$ means that some of the functions

$$D_j^{\pm}(\lambda) = \langle\omega_j^{\pm}|D^{\pm}(\lambda)|\omega_j^{\pm}\rangle = \exp(d_j^{\pm}(\lambda))$$

will have zeros and poles at $\lambda_k^{\pm}$, for more details see [8, 25]. Equation (6.7) can be easily rewritten in the form

$$\xi^{+}(x,\lambda) = \xi^{-}(x,\lambda)G(x,\lambda) \qquad G(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx}G_0(\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx}. \tag{6.8}$$

Then (6.8) together with

$$\lim_{\lambda\to\infty}\xi^{\pm}(x,\lambda) = \mathbb{1} \tag{6.9}$$

can be considered as a RHP with canonical normalization condition.

The solution $\xi^{+}(x,\lambda)$, $\xi^{-}(x,\lambda)$ to (6.8), (6.9) is called regular if $\xi^{+}(x,\lambda)$ and $\xi^{-}(x,\lambda)$ are nondegenerate and non-singular functions of λ for all $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ respectively. To the class of regular solutions of RHP there correspond Lax operators (1.2) without discrete eigenvalues. The presence of discrete eigenvalues $\lambda_k^{\pm}$ leads to singular solutions of the RHP; their explicit construction can be done by the Zakharov–Shabat dressing method [14, 20].

If the potential $q(x,t)$ of the Lax operator (1.2) satisfies the N-wave equation (1.1) then $S^{\pm}(t,\lambda)$ and $T^{\pm}(t,\lambda)$ satisfy the linear evolution equations

$$\mathrm{i}\frac{\mathrm{d}S^{\pm}}{\mathrm{d}t} - \lambda[I, S^{\pm}(t,\lambda)] = 0 \qquad \mathrm{i}\frac{\mathrm{d}T^{\pm}}{\mathrm{d}t} - \lambda[I, T^{\pm}(t,\lambda)] = 0 \tag{6.10}$$

while the functions $D^{\pm}(\lambda)$ are time independent. In other words $D_j^{\pm}(\lambda)$ can be considered as the generating functions of the integrals of motion of (1.1).

If we now impose a reduction on L it will reflect also on the scattering data. It is not difficult to check that if L satisfies (2.6) then the scattering matrix will satisfy

$$C_k(T(\Gamma_k(\lambda))) = T(\lambda) \qquad \lambda \in \mathbb{R}. \tag{6.11}$$

Note that strictly speaking (6.11) is valid only for real values of λ (more generally, for λ on the continuous spectrum of L). If we choose reductions with automorphisms of the form (2.1)–(2.3) for the FAS and for the Gauss factors $S^{\pm}(\lambda)$, $T^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ we will get

$$\begin{aligned} &S^{+}(\lambda) = A_1(\hat{S}^{-}(\lambda^*))^{\dagger}A_1^{-1} \qquad T^{+}(\lambda) = A_1(\hat{T}^{-}(\lambda^*))^{\dagger}A_1^{-1} \\ &D^{+}(\lambda) = (\hat{D}^{-}(\lambda^*))^{*} \qquad F(\lambda) = (F(\lambda^*))^{*} \qquad \eta = 1 \end{aligned} \tag{6.12}$$

$$\begin{aligned} &S^{+}(\lambda) = A_2(\hat{S}^{-}(-\lambda))^{T}A_2^{-1} \qquad T^{+}(\lambda) = A_2(\hat{T}^{-}(-\lambda))^{T}A_2^{-1} \\ &D^{+}(\lambda) = \hat{D}^{-}(-\lambda) \qquad \eta = -1 \end{aligned} \tag{6.13}$$

$$\begin{aligned} &S^{+}(\lambda) = A_3(S^{-}(-\lambda^*))^{*}A_3^{-1} \qquad T^{+}(\lambda) = A_3(T^{-}(-\lambda^*))^{*}A_3^{-1} \\ &D^{+}(\lambda) = (D^{-}(-\lambda^*))^{*} \qquad F(\lambda) = (F(-\lambda^*))^{*} \qquad \eta = -1 \end{aligned} \tag{6.14}$$

where we also used the fact that A_i belong to the Cartan subgroup of $\mathfrak{g}$. Next we make use of the integral representations for $d_j^{\pm}(\lambda)$ allowing one to reconstruct them as analytic functions in their regions of analyticity $\mathbb{C}_{\pm}$. In the case of absence of discrete eigenvalues we have [8, 11]

$$\mathcal{D}_j(\lambda) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\mu}{\mu-\lambda}\ln\langle\omega_j|\hat{T}^+(\mu)T^-(\mu)|\omega_j\rangle \tag{6.15}$$

where ω_j and $|\omega_j\rangle$ are the jth fundamental weight of $\mathfrak{g}$ and the highest weight vector in the corresponding fundamental representation $R(\omega_j)$ of $\mathfrak{g}$. The function $\mathcal{D}_j(\lambda)$ is a piece-wise analytic function of λ equal to

$$\mathcal{D}_j(\lambda) = \begin{cases} d_j^+(\lambda) & \text{for} \quad \lambda\in\mathbb{C}_+ \\ (d_j^+(\lambda) - d_{j'}^-(\lambda))/2 & \text{for} \quad \lambda\in\mathbb{R} \\ -d_{j'}^-(\lambda) & \text{for} \quad \lambda\in\mathbb{C}_- \end{cases} \tag{6.16}$$

where $d_j^{\pm}(\lambda)$ were introduced in (6.5) and the index j' is related to j by $w_0(\alpha_j) = -\alpha_{j'}$. Here w_0 is the Weyl reflection that maps the highest weight in $R(\omega_j)$ into the lowest weight of $R(\omega_j)$, see [21].

The functions $\mathcal{D}_j(\lambda)$ can be viewed also as generating functions of the integrals of motion. Indeed, if we expand

$$\mathcal{D}_j(\lambda) = \sum_{k=1}^{\infty}\mathcal{D}_{j,k}\lambda^{-k} \tag{6.17}$$

and take into account that $D^{\pm}(\lambda)$ are time independent we find that $\mathrm{d}\mathcal{D}_{j,k}/\mathrm{d}t = 0$ for all $k = 1,\ldots,\infty$ and $j = 1,\ldots,r$. Moreover, it can be checked that $\mathcal{D}_{j,k}$ expressed as functional of $q(x,t)$ has a kernel that is local in q, i.e. depends only on q and its derivatives with respect to x.

From (6.15) and (6.12)–(6.14) we easily obtain the effect of the reductions on the set of integrals of motion; namely, for the reduction (6.12)

$$\mathcal{D}_j(\lambda) = -\mathcal{D}_j^*(\lambda^*) \qquad \text{i.e.} \qquad \mathcal{D}_{j,k} = (-1)^{k+1}\mathcal{D}_{j,k}^* \tag{6.18}$$

with $\eta = 1$; for (6.13) we have

$$\mathcal{D}_j(\lambda) = -\mathcal{D}_j(-\lambda) \qquad \text{i.e.} \qquad \mathcal{D}_{j,k} = (-1)^{k+1}\mathcal{D}_{j,k} \tag{6.19}$$

and for (6.14)

$$\mathcal{D}_j(\lambda) = \mathcal{D}_j^*(-\lambda^*) \qquad \text{i.e.} \qquad \mathcal{D}_{j,k} = (-1)^{k}\mathcal{D}_{j,k}^*. \tag{6.20}$$

In particular, from (6.19) it follows that all integrals of motion with even k become degenerate, i.e. $\mathcal{D}_{j,2k} = 0$. The reductions (6.18) and (6.20) mean that 'half' of the integrals $\mathcal{D}_{j,2k}$ become real and the other 'half' $\mathcal{D}_{j,2k}$, purely imaginary.

We finish this section with a few comments on the simplest local integrals of motion. To this end we write down the first two types of integrals of motion $\mathcal{D}_{j,1}$ and $\mathcal{D}_{j,2}$ as functionals of the potential Q of (1.2). Skipping the details (see [8]) we get

$$\mathcal{D}_{j,1} = -\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty}\mathrm{d}x\,\langle[J,Q],[H_j^{\vee},Q]\rangle \tag{6.21}$$

and

$$\mathcal{D}_{j,2} = -\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\,\langle Q,[H_j^{\vee},Q_x]\rangle - \frac{\mathrm{i}}{3}\int_{-\infty}^{\infty}\mathrm{d}x\,\langle[J,Q],[Q,[H_j^{\vee},Q]]\rangle \tag{6.22}$$

where $H_j^{\vee} = 2H_{\omega_j}/(\alpha_j,\alpha_j)$.

The fact that $\mathcal{D}_{j,1}$ are integrals of motion for $j = 1, \ldots, r$, can be considered as a natural analogue of the Manley–Rowe relations [1,3]. In the case when the reduction is of the type (2.1), i.e. $p_\alpha = s_\alpha q_\alpha^*$, then (6.21) is equivalent to

$$\sum_{\alpha>0} \frac{2(\vec{a}, \alpha)(\omega_j, \alpha)}{(\alpha, \alpha)} \int_{-\infty}^{\infty} \mathrm{d}x\, s_\alpha |q_\alpha(x)|^2 = \mathrm{const} \tag{6.23}$$

and can be interpreted as relations between the densities $|q_\alpha|^2$ of the 'particles' of type α. For the other types of reductions such interpretation is not so obvious.

The integrals of motion $\mathcal{D}_{j,2}$ are directly related to the Hamiltonian of the N-wave equations (1.1), namely

$$H_{N\text{-wave}} = -\sum_{j=1}^{r} \frac{2(\alpha_j, \vec{b})}{(\alpha_j, \alpha_j)} \mathcal{D}_{j,2} = \frac{1}{2\mathrm{i}} \langle\langle \dot{\mathcal{D}}(\lambda), F(\lambda) \rangle\rangle_0 \tag{6.24}$$

where $\dot{\mathcal{D}}(\lambda) = \mathrm{d}\mathcal{D}/\mathrm{d}\lambda$ and $F(\lambda) = \lambda I$ is the dispersion law of the N-wave equation (1.1). In (6.24) we used just one of the hierarchy of scalar products in the Kac–Moody algebra $\hat{\mathfrak{g}} \equiv \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$:

$$\langle\langle X(\lambda), Y(\lambda) \rangle\rangle_k = \operatorname{Res} \lambda^{k+1} \langle \hat{D}^+(\lambda) X(\lambda), Y(\lambda) \rangle \qquad X(\lambda), Y(\lambda) \in \hat{\mathfrak{g}} \tag{6.25}$$

see [24].

7. Hamiltonian structures of the reduced N-wave equations

The generic N-wave interactions (i.e., prior to any reductions) possess a hierarchy of Hamiltonian structures. As mentioned in the introduction, the simplest one is $\{H^{(0)}, \Omega^{(0)}\}$; the symplectic form $\Omega^{(0)}$ after simple rescaling

$$q_\alpha \to \frac{q'_\alpha}{\sqrt{(a, \alpha)}} \qquad p_\alpha \to \frac{p'_\alpha}{\sqrt{(a, \alpha)}} \qquad \alpha \in \Delta_+$$

becomes canonical with q'_α being canonically conjugated to p'_α.

The hierarchy of symplectic forms is generated by the so-called generating (or recursion) operator $\Lambda = (\Lambda_+ + \Lambda_-)/2$:

$$\begin{gathered} \Lambda_\pm Z(x) = \operatorname{ad}_J^{-1} \left(\mathrm{i}\frac{\mathrm{d}Z}{\mathrm{d}x} + P_0 \cdot ([q(x), Z(x)]) + \mathrm{i}[q(x), I_\pm (\mathbb{1} - P_0)\,[q(y), Z(y)]] \right) \\ P_0 S \equiv \operatorname{ad}_J^{-1} \cdot \operatorname{ad}_J \cdot S \qquad (I_\pm S)(x) \equiv \int_{\pm\infty}^{x} \mathrm{d}y\, S(y) \end{gathered} \tag{7.1}$$

as follows:

$$\Omega^{(k)} = \frac{\mathrm{i}c_k}{2} \int_{-\infty}^{\infty} \mathrm{d}x\, \langle [J, \delta Q(x,t)] \underset{'}{\wedge} \Lambda^k \delta Q(x,t) \rangle \tag{7.2}$$

where $q(x,t) = [J, Q(x,t)]$. Using the completeness relation for the 'squared' solutions which is directly related to the spectral decomposition of Λ we can recalculate $\Omega^{(k)}$ in terms of the scattering data of L with the result

$$\begin{gathered} \Omega^{(k)} = \frac{c_k}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \lambda^k \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right) \\ \Omega_0^\pm(\lambda) = \langle \hat{D}^\pm(\lambda) \hat{T}^\mp(\lambda) \delta T^\mp(\lambda) D^\pm(\lambda) \underset{'}{\wedge} \hat{S}^\pm(\lambda) \delta S^\pm(\lambda) \rangle. \end{gathered} \tag{7.3}$$

The first consequence of (7.3) is that the kernels of $\Omega^{(k)}$ differs only by the factor λ^k; i.e., all of them can be cast into canonical form simultaneously. This is quite compatible with the results of [1, 2, 9] for the action-angle variables.

Again it is not difficult to find how the reductions influence $\Omega^{(k)}$. Using the invariance of the Killing form, from (7.3) and (6.12)–(6.14) we get respectively

$$\Omega_0^+(\lambda) = \left(\Omega_0^-(\lambda^*)\right)^* \tag{7.4}$$

$$\Omega_0^+(\lambda) = \Omega_0^-(-\lambda) \tag{7.5}$$

$$\Omega_0^+(\lambda) = \left(\Omega_0^-(-\lambda^*)\right)^*. \tag{7.6}$$

Then for $\Omega^{(k)}$ from (6.13) we find

$$\Omega^{(k)} = (-1)^{k+1}\Omega^{(k)}. \tag{7.7}$$

Like for the integrals $\mathcal{D}_{j,k}$ we find that the reductions (6.12) and (6.14) mean that $\Omega^{(k)}$ become real with a convenient choice for c_k.

Let us now briefly analyse the reduction (6.13) which may lead to degeneracies. We have already mentioned that $\mathcal{D}_{j,2k} = 0$, see (6.19); in addition, from (7.7) it follows that $\Omega^{(2k)} \equiv 0$. In particular, this means that the canonical 2-form $\Omega^{(0)}$ is also degenerate, so the N-wave equations with the reduction (6.13) do not allow Hamiltonian formulation with canonical Poisson brackets. However, they still possess a hierarchy of Hamiltonian structures:

$$\Omega^{(k)}\left(\frac{\mathrm{d}q}{\mathrm{d}t}, \cdot\right) = \nabla H^{(k)} \tag{7.8}$$

where $\nabla H^{(k)} = \Lambda\nabla_q H^{(k-1)}$; by definition $\nabla_q H = (\delta H)/(\delta q^T(x,t))$. Thus we find that while the choices $\left\{\Omega^{(2k)}, H^{(2k)}\right\}$ for the N-wave equations are degenerate, the choices $\left\{\Omega^{(2k+1)}, H^{(2k+1)}\right\}$ provide us with correct nondegenerate (though non-canonical) Hamiltonian structures, see [8, 10, 11].

This well known procedure for constructing the FAS of the Lax operators applies to the generic case when J has real and pair-wise distinct eigenvalues; such is the situation, for example, in the examples 14 ($\eta = 1$), 34 and 46.

However, in several other examples J has complex pair-wise distinct eigenvalues. In such cases one should follow the procedure described in [11]. We do not have the space to do so here, but will mention the basic differences. The most important one is that now the continuous spectrum Γ_L of L is not restricted to the real axis, but fills up a set of rays $\Gamma_L \equiv \cup_\alpha l_\alpha$ which are determined by $l_\alpha \equiv \{\lambda : \operatorname{Im}\lambda(\alpha, \vec{a}) = 0\}$. Then it is possible to generalize the procedure described above and to construct a fundamental analytic solution $\chi_\nu(x,\lambda)$ in each of the sectors closed between two neighbouring rays l_ν. Then we can again formulate a RHP, only now we will have sewing function determined upon each of the rays l_ν; possible discrete eigenvalues will lie inside the sectors.

If we now impose the reduction, the first consequence will be the symmetry of Γ_L with respect to it; more precisely, if $\lambda \in \Gamma_L$ then also $\kappa(\lambda) \in \Gamma_L$. Finally, we just note that the consequences of imposing the reductions (6.12)–(6.14) will be similar to the ones already described. In particular, the reduction (6.13) leads to the degeneracy of 'half' of the Hamiltonian structures, while the reductions (6.12) and (6.14) make these structures real with appropriate choices for c_k.

The last most difficult situation that takes place in many examples above arises when two or more of the eigenvalues of J become equal. Then the construction of the FAS requires the use of the generalized Gauss decompositions in which the factors $D^\pm(\lambda)$ are block-diagonal matrices while $T^\pm(\lambda)$ and $S^\pm(\lambda)$ are block-triangular matrices, see [19]. These problems will be addressed in subsequent papers.

8. Conclusions

We have described the systems of N-wave type related to the low-rank simple Lie algebras. In section 4 for any equivalence class of the Weyl group for the corresponding Lie algebra we chose one representative and we write down the corresponding reduced N-wave system. The complete list of the reduced systems is given in the tables below; two of the examples which we denote by ‡ can formally be listed in different locations of these tables, see remark 1.

$\mathfrak{g} \simeq \boldsymbol{A}_2$	$1\!\!1$	$\mathbb{Z}_2$	$\mathrm{Ad}_{\mathfrak{h}}$
$C(U^T(\pm\lambda)) = -U(\lambda)$	Ex. 1		
$C(U^*(\pm\lambda^*)) = -U(\lambda)$	Ex. 2	Ex. 3	
$C(U^\dagger(\pm\lambda^*)) = U(\lambda)$	Ex. 4	Ex. 5	Ex. 6

$\mathfrak{g} \simeq \boldsymbol{C}_2$	$1\!\!1$	$-1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathrm{Ad}_{\mathfrak{h}}$
$C(U^*(\pm\lambda^*)) = -U(\lambda)$	Ex. 7	Ex. 8‡	Ex. 10	Ex. 11	
$C(U(\pm\lambda)) = U(\lambda)$		Ex. 12			
$C(U^\dagger(\pm\lambda^*)) = U(\lambda)$	Ex. 8‡				Ex. 9

$\mathfrak{g} \simeq \boldsymbol{G}_2$	$1\!\!1$	$-1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathrm{Ad}_{\mathfrak{h}}$
$C(U^T(\pm\lambda)) = -U(\lambda)$	Ex.13				
$C(U^*(\pm\lambda^*)) = -U(\lambda)$		Ex. 14‡	Ex. 16	Ex. 17	
$C(U^\dagger(\pm\lambda^*)) = U(\lambda)$	Ex. 14‡				Ex. 15

$\mathfrak{g} \simeq \boldsymbol{A}_3$	$1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathrm{Ad}_{\mathfrak{h}}$
$C(U(\pm\lambda)) = U(\lambda)$		Ex. 18		
$C(U^*(\pm\lambda^*)) = -U(\lambda)$	Ex. 19	Ex. 20	Ex. 21	
$C(U^T(\pm\lambda)) = -U(\lambda)$	Ex. 22	Ex. 23	Ex. 24	
$C(U^\dagger(\pm\lambda^*)) = U(\lambda)$	Ex. 25	Ex. 27	Ex. 28	Ex.26

$\mathfrak{g} \simeq \boldsymbol{B}_3$	$1\!\!1$	$-1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_2^{(4)}$	$\mathrm{Ad}_{\mathfrak{h}}$
$C(U(\pm\lambda)) = U(\lambda)$		Ex. 29	Ex. 30	Ex. 31	Ex. 32	Ex. 33	
$C(U^\dagger(\pm\lambda^*)) = U(\lambda)$	Ex. 34	Ex. 36	Ex. 37	Ex. 38	Ex. 39	Ex. 40	Ex. 35

$\mathfrak{g} \simeq \boldsymbol{C}_3$	$1\!\!1$	$-1\!\!1$	$\mathbb{Z}_2^{(1)}$	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_2^{(4)}$	$\mathrm{Ad}_{\mathfrak{h}}$
$C(U(\pm\lambda)) = U(\lambda)$		Ex. 41	Ex. 42	Ex. 43	Ex. 44	Ex. 45	
$C(U^\dagger(\pm\lambda^*)) = U(\lambda)$	Ex. 46	Ex. 48	Ex. 49	Ex. 50	Ex. 51	Ex. 52	Ex. 47

The N-wave systems related to reductions from the same equivalence class will be equivalent. The empty boxes in the tables above mean that the N-wave system after the reduction becomes trivial.

We end this paper with several remarks.

(1) The $\mathbb{Z}_2$-reductions which act on λ by $\Gamma_1(\lambda) = \lambda^*$, combined with Cartan involutions on $\mathfrak{g}$ lead in fact to restricting of the system to a specific real form of the algebra $\mathfrak{g}$.

(2) To all reduced systems given above we can apply the analysis in [8, 11] and derive the completeness relations for the corresponding systems of 'squared' solutions. Such analysis will allow one to prove the pair-wise compatibility of the Hamiltonian structures and eventually to derive their action-angle variables, see [1, 9] for the $\boldsymbol{A}_n$-series.

(3) These results can be extended naturally in several directions:

- for NLEE with other dispersion laws. This would allow us to study the reductions of the multicomponent NLS-type equations (see [26]), Toda type systems, etc.

- for Lax operators with more complicated λ-dependence, for example

$$L(\lambda)\psi = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + U_0(x,t) + \lambda U_1(x,t) + \frac{1}{\lambda}U_{-1}(x,t)\right)\psi(x,t,\lambda) = 0.$$

This would allow us to investigate more complicated reduction groups as, for example, $\mathbb{T}$, $\mathbb{O}$ (see [27]) and the possibilities to embed them as subgroups of the Weyl group of $\mathfrak{g}$.

(4) In a series of papers Calogero [28] demonstrated that a number of integrable NLEE including ones of N-wave type are universal and widely applicable in physics. Although the examples we have here do not seem to coincide with the ones in [28] there is a hope that some of them might find applications in physics.

(5) Some preliminary results concerning the $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction group are reported in [29].

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Appendix A

For each of the algebras used above we list the sets $\mathcal{M}_\alpha$ which consist of all pairs of roots β, γ such that $\beta + \gamma = \alpha$. The roots are denoted as usual by j, k or i, j, k; the negative roots are overlined. Obviously $\mathcal{M}_\alpha$ describes all possible decays of the α-type wave.

Next we write in more compact form the quantities ω_{jk} (1.9) using the following notation:

$$\kappa_1 = a_2b_3 - a_3b_2 \qquad \kappa_2 = a_3b_1 - a_1b_3 \qquad \kappa_3 = a_1b_2 - a_2b_1. \tag{A.1}$$

(1) $\mathfrak{g} \simeq$ ***A_2*-algebra.** For this algebra there is only one 'decay' of roots: (11) = (10) + (01) and the corresponding coefficient is $\omega_{10,01} = 6\kappa$, where $\kappa = a_1b_2 - a_2b_1$.

(2) $\mathfrak{g} \simeq$ ***C_2*-algebra.** Here there are two 'decays':

$$(21) = (11) + (10) \qquad (11) = (10) + (01)$$

and the corresponding coefficients ω_{jk} are:

$$\omega_{11,10} = -\omega_{10,01} = 2\kappa \qquad \kappa = a_1b_2 - a_2b_1. \tag{A.2}$$

(3) $\mathfrak{g} \simeq$ ***G_2*-algebra.** The sets $\mathcal{M}_\alpha$ are as follows:

$$\begin{aligned} &\mathcal{M}_{10} = \{(11,\overline{1}), (21,\overline{11}), (31,\overline{21})\} &&\mathcal{M}_{1} = \{(11,\overline{10}), (32,\overline{31})\} \\ &\mathcal{M}_{11} = \{(10,1), (21,\overline{10}), (32,\overline{21})\} &&\mathcal{M}_{21} = \{(10,11), (31,\overline{10}), (32,\overline{11})\} \\ &\mathcal{M}_{31} = \{(21,10), (32,\overline{1})\} &&\mathcal{M}_{32} = \{(31,1), (11,21)\} \end{aligned} \tag{A.3}$$

and the coefficients ω_{jk} without reductions are

$$\omega_{10,01} = -2\omega_{10,11} = 2\omega_{21,10} = 2\omega_{31,01} = -2\omega_{21,11} = 6\kappa. \tag{A.4}$$

(4) $\mathfrak{g} \simeq$ ***A_3*-algebra.** The sets $\mathcal{M}_\alpha$ for this algebra are as follows:

$$\begin{aligned} &\mathcal{M}_{100} = \{(110,\overline{10}), (111,\overline{11})\} &&\mathcal{M}_{10} = \{(110,\overline{100}), (11,\overline{1})\} \\ &\mathcal{M}_{1} = \{(11,\overline{10}), (111,\overline{110})\} &&\mathcal{M}_{110} = \{(100,10), (111,\overline{1})\} \\ &\mathcal{M}_{11} = \{(10,1), (111,\overline{100})\} &&\mathcal{M}_{111} = \{(11,100), (110,1)\} \end{aligned} \tag{A.5}$$

and the general form of the quantities ω_{jk} are

$$\begin{aligned} &\omega_{100,010} = 2(\kappa_1 + \kappa_2 + \kappa_3) &&\omega_{010,001} = 2(3\kappa_1 - \kappa_2 - \kappa_3) \\ &\omega_{110,001} = 2(\kappa_1 - 3\kappa_2 + \kappa_3) &&\omega_{100,011} = 2(-\kappa_1 - \kappa_2 + 3\kappa_3). \end{aligned} \tag{A.6}$$

(5) $\mathfrak{g} \simeq B_3$**-algebra.** The sets $\mathcal{M}_\alpha$ here are given by

$$\begin{aligned}
\mathcal{M}_{100} &= \{(110, \overline{10}), (111, \overline{11}), (112, \overline{12})\} \\
\mathcal{M}_{10} &= \{(110, \overline{100}), (11, \overline{1}), (122, \overline{112})\} \\
\mathcal{M}_{1} &= \{(11, \overline{10}), (12, \overline{11}), (111, \overline{110}), (112, \overline{111})\} \\
\mathcal{M}_{110} &= \{(100, 10), (111, \overline{1}), (122, \overline{12})\} \\
\mathcal{M}_{11} &= \{(1, 10), (12, \overline{11}), (111, \overline{100}), (122, \overline{111})\} \\
\mathcal{M}_{111} &= \{(100, 11), (110, 1), (122, \overline{11}), (112, \overline{1})\} \\
\mathcal{M}_{12} &= \{(1, 11), (112, \overline{100}), (122, \overline{110})\} \\
\mathcal{M}_{112} &= \{(100, 12), (111, 1), (122, \overline{10})\} \\
\mathcal{M}_{122} &= \{(11, 111), (112, 10), (110, 12)\}
\end{aligned} \tag{A.7}$$

and the coefficients ω_{jk} (1.9) are

$$\begin{aligned}
&\omega_{100,10} = 2(\kappa_1 + \kappa_2 + \kappa_3) \qquad \omega_{100,11} = 4\kappa_3 \qquad \omega_{100,12} = -2(\kappa_1 + \kappa_2 + \kappa_3) \\
&\omega_{110,1} = 2(\kappa_1 - \kappa_2 + \kappa_3) \qquad \omega_{10,1} = 4\kappa_1 \qquad \omega_{10,112} = -2(\kappa_1 - \kappa_2 - \kappa_3) \\
&\omega_{1,110} = -4\kappa_2 \qquad \omega_{1,11} = -4\kappa_1 \qquad \omega_{1,111} = -4\kappa_2 \qquad \omega_{11,111} = -4\kappa_3
\end{aligned} \tag{A.8}$$

with κ_1, κ_2 and κ_3 given by (A.1).

(6) $\mathfrak{g} \simeq C_3$**-algebra.** Here we have

$$\begin{aligned}
\mathcal{M}_{100} &= \{(110, \overline{10}), (111, \overline{11}), (121, \overline{21}), (221, \overline{121})\} \\
\mathcal{M}_{10} &= \{(110, \overline{100}), (11, \overline{1}), (21, \overline{11}), (121, \overline{111})\} \\
\mathcal{M}_{1} &= \{(11, \overline{10}), (111, \overline{110})\} \\
\mathcal{M}_{110} &= \{(100, 10), (111, \overline{1}), (121, \overline{11}), (221, \overline{111})\} \\
\mathcal{M}_{11} &= \{(1, 10), (111, \overline{100}), (21, \overline{10}), (121, \overline{110})\} \\
\mathcal{M}_{111} &= \{(100, 11), (110, 1), (221, \overline{110}), (121, \overline{10})\} \\
\mathcal{M}_{21} &= \{(10, 11), (121, \overline{100})\} \\
\mathcal{M}_{121} &= \{(100, 21), (110, 11), (111, 10), (221, \overline{100})\} \\
\mathcal{M}_{122} &= \{(121, 100), (110, 111)\}
\end{aligned} \tag{A.9}$$

and the coefficients ω_{jk} (1.9) are

$$\begin{aligned}
&\omega_{100,10} = 2(\kappa_1 + \kappa_2 + \kappa_3) \qquad \omega_{100,11} = -2(\kappa_1 + \kappa_2 - \kappa_3) \qquad \omega_{100,21} = 4\kappa_3 \\
&\omega_{110,11} = -2(\kappa_1 - \kappa_2 + \kappa_3) \qquad \omega_{10,11} = -4\kappa_1 \qquad \omega_{1,110} = -4\kappa_2 \\
&\omega_{10,1} = 4\kappa_1 \qquad \omega_{10,111} = -2(\kappa_1 - \kappa_2 - \kappa_3) \\
&\omega_{100,121} = -4\kappa_3 \qquad \omega_{110,111} = -4\kappa_2
\end{aligned} \tag{A.10}$$

where again κ_1, κ_2 and κ_3 are given by (A.1).

Appendix B

Here we show some typical interaction terms for each of the four types of reductions. There we have also replaced the constant c_0 by its appropriate value. The form of these terms crucially depends on whether the roots α, β and γ belong to Δ_+^0 or Δ_+^1.

(a) $\alpha, \beta, \gamma \in \Delta_+^0$:

$$(1) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x \, (q_\alpha q_{\beta'}^* q_{\gamma'}^* + \eta q_{\alpha'}^* q_\beta q_\gamma), \tag{B.1}$$

$$(2) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x \, (q_\alpha q_{\beta'} q_{\gamma'} + \eta q_{\alpha'} q_\beta q_\gamma), \tag{B.2}$$

$$(3) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{-\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p^*_{-\beta'} p^*_{-\gamma'} - \eta p^*_{-\alpha'} q_\beta q_\gamma), \tag{B.3}$$

$$(4) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p_{-\beta'} p_{-\gamma'} - \eta p_{-\alpha'} q_\beta q_\gamma). \tag{B.4}$$

(*b*) $\alpha, \gamma \in \Delta_+^0, \beta \in \Delta_+^1$:

$$(1) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p^*_{-\beta'} q^*_{\gamma'} + \eta q^*_{\alpha'} q_\beta q_\gamma), \tag{B.5}$$

$$(2) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p_{-\beta'} q_{\gamma'} + \eta q_{\alpha'} q_\beta q_\gamma), \tag{B.6}$$

$$(3) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{-\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha q^*_{-\beta'} p^*_{-\gamma'} - \eta p^*_{-\alpha'} q_\beta q_\gamma), \tag{B.7}$$

$$(4) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha q_{-\beta'} p_{-\gamma'} - \eta p_{\alpha'} q_\beta q_\gamma). \tag{B.8}$$

The right-hand side of (B.5) coincides with the standard interaction terms in (3.10) only if $\beta' = \beta$ and $\gamma' = -\gamma$, i.e. only if $\beta, \gamma \in \Delta_+^\perp$. There are a special subcases of (B.1) when $\gamma' = \alpha$; then

$$(1) \quad \left\{ \begin{matrix} \alpha \\ \alpha', \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (|q_\alpha|^2 p^*_{-\beta'} + \eta |q_{\alpha'}|^2 q_\beta), \tag{B.9}$$

$$(2) \quad \left\{ \begin{matrix} \alpha \\ \alpha', \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha^2 p_{-\beta} + \eta q_{\alpha'}^2 q_\beta), \tag{B.10}$$

$$(3) \quad \left\{ \begin{matrix} \alpha \\ \alpha', \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{-\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p^*_{-\alpha} q^*_{-\beta'} - \eta p_{-\alpha'} q^*_{\alpha'} q_\beta), \tag{B.11}$$

$$(4) \quad \left\{ \begin{matrix} \alpha \\ \alpha', \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p_{-\alpha} q_\beta - \eta p_{-\alpha'} q_{\alpha'} q_\beta). \tag{B.12}$$

(*c*) $\alpha, \beta, \gamma \in \Delta_+^1$,:

$$(1) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p^*_{-\beta'} p^*_{-\gamma'} + \eta p^*_{-\alpha'} q_\beta q_\gamma), \tag{B.13}$$

$$(2) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p_{-\beta'} p_{-\gamma'} + \eta p_{-\alpha'} q_\beta q_\gamma), \tag{B.14}$$

$$(3) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{-\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha q^*_{-\beta'} q^*_{-\gamma'} - \eta q^*_{-\alpha'} q_\beta q_\gamma), \tag{B.15}$$

$$(4) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha q_{-\beta'} q_{-\gamma'} - \eta q_{-\alpha'} q_\beta q_\gamma). \tag{B.16}$$

(*d*) $\alpha, \gamma \in \Delta_+^1, \beta \in \Delta_+^0$:

$$(1) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha q^*_{\beta'} p^*_{-\gamma'} + \eta p^*_{-\alpha'} q_\beta q_\gamma), \tag{B.17}$$

$$(2) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha q_{\beta'} p_{-\gamma'} + \eta p_{-\alpha'} q_\beta q_\gamma), \tag{B.18}$$

$$(3) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = \frac{1}{\sqrt{-\eta}} n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p^*_{-\beta'} q^*_{-\gamma'} - \eta q^*_{-\alpha'} q_\beta q_\gamma), \tag{B.19}$$

$$(4) \quad \left\{ \begin{matrix} \alpha \\ \beta, \gamma \end{matrix} \right\}_R = n_{\alpha,\alpha'} \omega_{\beta,\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, (q_\alpha p_{-\beta'} q_{-\gamma'} - \eta q_{-\alpha'} q_\beta q_\gamma), \tag{B.20}$$

with a similar particular case when $\alpha = -\gamma'$.

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